Journal of Scientific Exploration, Vol. 13, No. 2, pp. 177–188, 1999 0892-3310/99

Variations on the Foundations of Dirac's Quantum Physics

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Abstract — The Dirac algebra is examined as a hypercomplex number system, where there are six basic, anti-automorphic conjugations. However, we can concentrate on *only half* of this algebra and there find *all* the old physics that Dirac found. We do not even need to introduce matrices at all (which is a surprise to the field theory community). The Dirac algebra can be readily generalized using quaternions to expand the system, and Dirac's old equation is also generalized by introducing a new, *multi-mass* part. Mass may actually be *very complicated* at the quark level, where we never directly see the particles' tracks.

Keywords: Dirac's equation — Lorentz's group — Pauli's algebra — spin — wave equations

Introduction

The deepest insights we have reached, in understanding the design of our world [1], [2], [3], [7], [8], have proven to be associated with rather sophisticated number systems. This math/physics world has been inaccessible to those who have not studied the mathematics called matrices. As a result, even the physics books for junior/senior physics *majors* avoid this deep coverage.

It may be hopeless, but this paper is an attempt to open this world to non-physicists, and also to many physicists who never got this far in their studies.

The Number Systems

The core structure here is a generalization of the complex number (hypercomplex number) system. These complex numbers have two parts: $A = a(1) + b(i)$ where $(i)(i) \equiv -(1)$. The *basis* elements here, $\{(1), (i)\}$, define the system. We can change the number representation to $\{(e_0), (ie_0)\}$ and then $(ie_0)(ie_0) \equiv -(e_0)$. We can also write the same complex number as $A = a^0(e_0) + b^0(ie_0)$, where a^0 and b^0 are any real numbers. There is a third form: $A = a^0(e_0) + b^1(-ie_1)$, where $(-ie_1)(-ie_1) \equiv -(e_0)$. This still means, for example, $3 + 2i \leftrightarrow 3(e_0) + 2(-ie_1)$. Is this much ado about nothing? The reason for this complicated looking form is that it generalizes nicely to four-part numbers, which were invented over the decade preceding 1843 and published in 1844, by R. Hamilton. He invented the *equivalent* of $\{(e_0), (-ie_1), (-ie_2), (-ie_3)\}$, with the *new rules* $(-ie_1)(-ie_2) \equiv +(-ie_3) \equiv -(-ie_2)(-ie_1)$, and cyclic — like going around a triangle: ccw gives + and cw gives – . The new elements, $(-ie_2)$ and $(-ie_3)$, are like $(-ie_1)$ in that their squares are $-(e_0)$. He called this system the quaternions. He wrote two large volumes on calculus, using quaternions, and felt they were the key to the universe. No one taught me about this, all through school to the Ph.D. level! It should be introduced in pre-college math classes as just a curiosity (but a beautiful one).

There is more rich history that we must skip over here [4], [5], [6], [9], but W. Pauli, in 1927, *finally* advanced the use of these 1843 number ideas, by effectively (though he did not know it) doubling Hamilton's number system to the H-P number system with an eight-basis: $\{(e_0), (-ie_1), (-ie_2), (-ie_3); (ie_0), (e_1), (e_2), (e_3)\}$. We must now define the $8 \times 8 = 64$ element multiplication table. It is simple enough that all 64 combinations can be done in our heads. The details are not important here. The famous Pauli pattern started out with $(e_1)(e_2) \equiv (ie_3) \equiv -(e_2)(e_1)$, and cyclic permutations. Also, $(e_1)(e_1) = (e_0)$, ... $(ie_0)(e_1) = (ie_1)$, ...

The complex numbers have the equivalent basis choices $\{(e_0), (ie_0)\}$, $\{(e_0), (-ie_1)\}$, $\{(e_0), (-ie_2)\}$, or $\{(e_0), (-ie_3)\}$. Complex numbers have a complex conjugation $[a(e_0) + b(ie_0)]^{\uparrow} \equiv [a(e_0) - b(ie_0)]$, which we can conveniently describe as $\{(e_0), (ie_0)\}^{\uparrow} \equiv \{+(e_0), -(ie_0)\}$. For the *quaternions,* we generalize this conjugation as expected: $\{(e_0), (-ie_1), (-ie_2), (-ie_3)\}^{\uparrow} \equiv \{(e_0), -(-ie_1), -(-ie_2), -(-ie_3)\}$. We define "$-(e) \equiv (-e)$" for convenience, but "$i(e)$" is meaningless *here*. It is tedious to prove, but $(Q_1Q_2)^{\uparrow} = Q_2^{\uparrow}Q_1^{\uparrow}$, in general, for the quaternions. If you know matrices, then these eight basis elements are "like" 2×2 complex matrices, and $(...)^{\uparrow}$ is hermitian conjugation — flip around the main diagonal and take the complex conjugate of each element. In matrices, $(AB)^{\text{conj}} = B^{\text{conj}}A^{\text{conj}}$ is generally true when conj ≡ hermitian conjugation (the only really *basic* conjugation for matrices). They have trace and determinant, which are missing in the hypercomplex numbers.

In hypercomplex numbers there are other conjugations, just as important. They mostly show up in other number systems larger than the quaternions, but the four-part system $\{(e_0), (-ie_1), (e_2), (e_3)\}$ is closed, and it has $\{...\}^{\uparrow} \equiv \{(e_0), -(-ie_1), +(e_2), +(e_3)\}$. Again, we can show that $(AB)^{\uparrow} \equiv B^{\uparrow}A^{\uparrow}$ here. We "can imagine" the very useful thought sequence $(-ie_1)(e_2) \rightarrow (-ie_1e_2) \rightarrow (-iie_3) \rightarrow +(e_3)$, so we *define* $(-ie_1)(e_2) \equiv +(e_3)$, using this "mental crutch."

The minimal number system for doing serious, relativistic physics is this eight-part, H-P system above. Here,

$$P \equiv P^{\uparrow} = P^0(e_0) + P^1(e_1) + P^2(e_2) + P^3(e_3) + 0(ie_0) + 0(ie_1) + 0(ie_2) + 0(ie_3) \tag{1}$$

It has four real number coefficients $\{P^0, P^1, P^2, P^3\}$. For short, we write this as $P = P^{\mu}(e_{\mu})$, sum on $\mu = 0,1,2,3$. This hypercomplex number P describes the energy and momentum of a classical particle moving in space and time, where we write $X \equiv x^{\mu}(e_{\mu}) = ct(e_0) + x(e_1) + y(e_2) + z(e_3)$, $\mu = 0,1,2,3$, so $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. The c here is light speed. (The notation is more complicated than the content — but once mastered, it is very convenient.) Physically, P and X are four-vectors. They "combine" space and time into one, elegant, mathematical thing. This is how time is considered as the *fourth* dimension. It is not really a *fourth* dimension, it *just acts like it,* mathematically, in the formulas that have proven to be useful for our world. To fit the full array of classical physics at the relativistic level, we *need a second conjugation:* $\{(e_{\mu}), (ie_{\mu})\}^\wedge \equiv \{(e_0), (ie_0), -(e_k), -(ie_k)\}$, $k = 1,2,3$. Again, we can show that $(AB)^\wedge = B^\wedge A^\wedge$ holds for any two, eight-part numbers in the H-P system. Mathematicians call this order reversal the anti-automorphic property. For $X \equiv x^{\mu}(e_{\mu})$, we see that $X^\wedge$ leaves the time part alone and inverts the (x, y, z) coordinates. This is called a space inversion or a *parity* transformation. This conjugation can next be used to invent the very important inner-product concept, defined by

$$\begin{aligned} P^\wedge P = K &= \left(P^0 P^0 - P^1 P^1 - P^2 P^2 - P^3 P^3\right)\left(e_0\right) \\ &= \left(P^0 P^0 - P^k P^k\right)\left(e_0\right) \end{aligned} \tag{2}$$

sum on $k = 1,2,3$. We next invent the standard notation $(P^{\mu}P_{\mu}) \equiv (P^0P^0 - P^kP^k)$, only for convenience. We also find that $X^\wedge X = \ldots = (x^{\mu}x_{\mu})(e_0) \equiv (x^0x^0 - x^kx^k)(e_0)$, as expected in *flat* space-time. There are NO surviving cross-terms here. For $X^{\uparrow}X$, there *would* be cross-terms. Try it yourself and see. This $X^{\uparrow}X$ pattern is NOT physically useful, it seems.

This quaternion (also called *symplectic*) conjugation also has the physical application of defining the electromagnetic field:

$$F = -E^k\left(e_k\right) + cB^k\left(ie_k\right) + 0\left(e_0\right) + 0\left(ie_0\right) = -F^\wedge \tag{3}$$

Again, c is light speed. Here, $\{E^k, B^k\}$ is the electromagnetic field component set, $\{E_x, E_y, E_z, B_x, B_y, B_z\}$. Moving charges are the source of this F field and these moving charges are described by the mathematical current four-vector $J = J^{\uparrow} = J^{\mu}(e_{\mu})$, which we do not have to go into further here. (See my book (Edmonds, 1997) or mcneese.edu/colleges/science/physics if interested.)

I cannot go on without pointing out the beauty of Maxwell's equation: $\partial F = J$, where ∂ is an "operator," $\partial^{\mu}(e_{\mu})$, involving partial derivatives on the F field components, which may not mean much to the reader. Clearly, $\partial F = J$ is *simple* and *elegant*; it is also true for our *real* world! There are eight distinct equations here, when this is expanded out. Maxwell found this law in the electrical data available in the 1800s. His was an inelegant set of equations, because Hamilton's system was *not* generalized as it *should have been*, for the mid- to late-1800s physics.

Natural Groups

Now we can deal with the natural groups in the H-P number system. A group has members, such as A and B, with some special property. Then $AB = C$ produces a third element C with the same property, from the specified “combination” of the members A and B, represented abstractly by AB. In general the combination rule takes many forms. There must be an A^{-1} element such that $AA^{-1} = 1 = A^{-1}A$. There must be an identity element which is essentially "1", or (e_0) in our case.

The basis elements of H-P form a finite group with eight members:.$\{(e_\mu), (ie_\mu)\}$. There are really 16 members here. We do not usually list "$-(e)$" as a distinct member here, but it really is.

The H-P system has two natural groups: $SS^\uparrow \equiv 1(e_0)$ and $LL^\wedge \equiv 1(e_0)$. The needed inverses clearly exist here and the identity is clearly $1(e_0)$. Here, any L in the group has the general form $L = a^\mu(e_\mu) + b^\mu(ie_\mu)$, sum on $\mu = 0,1,2,3$. There are eight parameters here in L, $\{a^\mu, b^\mu\}$, but $LL^\wedge = 1(e_0) = 1(e_0) + 0(ie_0)$. This means that two equations link the eight parameters, leaving only six free parameters. We say this is a Lie group. It has an infinity of members, so it is in a very different class from the finite group above. The other group here, $SS^\uparrow \equiv 1(e_0) = 1(e_0) + 0(e_1) + 0(e_2) + 0(e_3)$. This means that there are four equations limiting the eight parameters, so there are only four free parameters here.

The concept of generators is also useful in Lie groups. It turns out that (in this hypercomplex number language) we find the L generators from the requirement $(...)^\wedge = -(...)$ for the basis elements in the number system. For L, this gives the generator set $\{(ie_k), (e_k)\}$, $k = 1,2,3$. The squares of the generators are also useful. Here the squares give the pattern $\{- - - +++\}$. These signs dictate the structure of the one-parameter sub-groups in L. For example, we have $L_\theta = \cos\theta(e_0) + \sin\theta(ie_1)$ and, for small θ, this becomes $L_\theta \approx 1(e_0) + \theta(ie_1)$. This L, we say, is *close* to the identity, and $\theta\theta \approx 0$ has been used here. In general, $L \approx 1 + \varepsilon$ is any member of the group *close* to the identity. Then $LL^\wedge = 1$ leads to $\varepsilon + \varepsilon^\wedge = 0$. In the above case, $\varepsilon = \theta(ie_1)$, with generator (ie_1). For large θ, we still have $LL^\wedge = 1(e_0)$ exactly here, because the generator's square is negative. The other type of L member has those generators g with $gg = +(e_0)$. Then $L_\theta = \cosh\theta(e_0) + \sinh\theta(e_1)$, for example. Here we also have, for small parameter θ, $L \approx 1(e_0) + \theta(e_1)$. Physicists call these "sinh-type" L_θ transformation *boosts*, for historical reasons.

What is the importance of all this? *Where is the physics here?* That is a fair question. It turns out that group L is very important to the physics, and group S is not as important as L. So we never know where nature will take us. The S group can be shown to have the generators $\{(ie_k), (ie_0)\}$, with all four squares negative, so all S_θ members have *sine* (not *sinh*) type terms here. We say this is a *compact* group. S does not go to infinity here in this compact group, as θ

goes to infinity. This split into compact and non-compact groups turns out to be very important to the physics of the quark world, or so it seems *now*. The L group has three compact and three non-compact group parameters. It has the three-parameter sub-group $\{(ie_k)\}$, which is in common with a sub-group of S where $S^{\uparrow} = S^{\wedge}$. The group S has the fourth generator commute with the other three. This splits it off, in a sense, and the S group is called SU(2)$\otimes U(1)$ in matrix language. This symbolism is not important, but we shall use their old names. The L group is called SL(2,C) by matrix-loving mathematicians.

Classical Physics

With all this four-vector and group machinery, in the H-P number system, we are finally ready to *start our classical design of a universe.* These pieces are like the pigments that fill in the painting. We have shown the successful Maxwell equation. Like Lorentz, we can use it to find the basic symmetry in nature. The equation $\partial F = J$ has $\partial^{\uparrow} = \partial$, $F^{\wedge} = -F$, and $J^{\uparrow} = J$. Why does nature use this particular structure for the electromagnetic field? It turns out to be simple but that is not enough. There are too many possibilities that are also simple. We need more guidance than that. It seems 'God chose' the additional rule of *form covariance* under a special group. This takes the form $F \to F' \equiv L^{\wedge} FL$ because then $F'^{\wedge} = \ldots = -F'$. Pretty, you must admit. Similarly, we are then guided to choose $\partial \to \partial' = L^{\uparrow}\partial L$, so that $\partial'^{\uparrow} = \ldots = \partial'$. Clearly then, J is like ∂ and $J \to J' = L^{\uparrow}JL$. Notice that we do not have to define $J = J^{\uparrow}$ here. The left-hand side of Maxwell's eqation does not have $[\partial F]^{\uparrow} = [\partial F]$, so J does not have to either. If J is not simple like this, then we get the possibility of magnetic monopole J sources for the E and B fields in F. So, maybe nature chose *instead*

$$\partial F + \left[\partial F\right]^{\uparrow} = J \tag{4}$$

so that $J = J^{\uparrow}$ is now *necessary* and there are *no* monopoles in our universe. We are *guessing* in either case and only experiment can answer this, it seems. So far, no monopoles have been seen.

Now we can invent the game of covariance testing of equations as follows. Start with the *guess* $\partial F = J$ and then form $\partial' F' = J' = L^{\uparrow}\partial LL^{\wedge} FL = L^{\uparrow}JL$. We have some "trapped" L's here, so *we define* $LL^{\wedge} = 1(e_0)$. This allows us to multiply the above equation from the left by $L^{\uparrow\wedge}$ and from the right by $L^{\wedge}$, and see that all the L's can be canceled out. This is the SL(2,C) group above. $LL^{\wedge}$ fits naturally here, not $LL^{\uparrow}$. The sub-group where $L^{\wedge} = L^{\uparrow}$ is the SU(2) sub-group, in common with S. Why would not you, if you were God, use this *simpler* group, SU(2), for your covariance testing? It allows a lot more possible laws. The more complicated the L group, used in the design, the less equations that can pass through this *filter*. Thus there could be a much *larger reality* for the universe, with larger groups and larger curved space-time. Maxwell only shows us its smallest parts.

Where does Einstein fit in all of this? He does not very much. Lorentz got the right group from Maxwell. Einstein arrived at the *same* group a few years later, from totally different ideas having little to do with any previous *measurements* that had been made. He made philosophical guesses about "God's preferences" in the design of reality, like Aristotle guessed a lot using pure reason; but Einstein got lucky. His "wrong" guesses led very simply to the right covariance group, Lorentz's, with no hypercomplex numbers used at all. He did do something very important here that Lorentz missed. Finding the group L, in Maxwell's equation, does not tell you much about its usefulness. Einstein realized that this group relates, not just to an electron's shrinking and slowed time keeping, but to real measurements made by *real people* in the universe, if they are moving. He thus predicted that the *equivalent* of $X' = L^{\uparrow}XL$ means that the measured values of x,y,z,t for the *moving* observer (or a rotated and at *rest* observer) are *different* from those obtained by the *really at rest, non-rotating observer*. This has been confirmed as true for the real world. Notice that we got the right equation above, for X', without any consideration of moving observers. Indeed ∂' is just a part of the game, to find the *good equations* in nature. Once we have found them, then we must worry about *which* physical frame is the right one in which to see that these laws do indeed work in the lab, to confirm our predictions. That is a whole other issue, which was answered in 1965 with the discovery of the 3°K *Gamow radiation* left over from the Big Bang. We *know the right frame now,* and we have *measured* our real motion as about 2/1000 of the speed of light relative to *it* (moving toward the galaxy Hydra in Leo). The *American* and *European* physics teaching journals *both refused* to let me say just that, in print, in their journals. They defend the current faith just like the Pope did in Galileo's day.

Everyone liked Einstein's approach better than Lorentz's (except probably Lorentz). We favor philosophical principles. It is a hang up we need to get over in physics. We favored *parity* symmetry, *etc.,* also, and got burned there. The math ultimately will give us the laws, and we will *never* know, philosophically, why they are correct. We do not really understand relativity the way we thought we did. The books need to be changed. The length contraction and time slowing are real *because* the form covariance of Maxwell's equation reflects some deeper, unknown laws that have a certain L symmetry filter on their structure. When you are really moving, this *causes* you to really slow in time and shrink in the direction of motion. Somehow the space itself *does it* — yes, the *ether* is back with a vengeance. Going very very fast means that you are going very very fast with respect to this *ether* — it shrinks you and slows your time. *HOW*? No one presently knows, and we cannot know until we really know the *vacuum* state and its number of dimensions, *etc.* — all of its other properties. It is a sea of virtual stuff, and you must run this gauntlet, when moving — who knows what it can do to you!? We only know that it does little to individual sub-atomic particles, except to slow their internal time. Muons live longer, before they decay, when moving. *We have to stay humble and skeptical.*

Symmetry of form mainly helps us guess at new laws and equations. Besides Maxwell, in H-P, we also have Newton. How would you design classical mechanics within the H-P number system? There is no classical design for our real universe. There may be exact classical laws in other universes, but not ours. In our universe, there are *other* laws at a *much smaller scale size* and these dictate how the world winds up looking to huge creatures observing other huge objects, where all the subtle reality gets averaged away. Our quantum level physics seems to reduce to the following classical approximation. We have

$$\mathrm{d}P/\mathrm{d}\tau = v\,F + \left(vF\right)^{\uparrow} \tag{5}$$

where $P = P^{\uparrow} \equiv mv$, $v \equiv \mathrm{d}X/\mathrm{d}\tau$, $\mathrm{d}\tau$ is the time kept on the moving mass' clock. We have seen the F field above already. There is a missing coupling constant here, with charge q, that can be absorbed into the m (constant) parameter. Is this proposed equation form covariant under the group L? We *guess* that $m' = m$, $\tau' = \tau$ and $c' = c$ in this game. The test then starts as follows:

$$\begin{aligned} \mathrm{d}P'/\mathrm{d}\tau' &= v'\,F' + \left(v'F'\right)^{\uparrow} = \mathrm{d}\left(L^{\uparrow}PL\right)\Big/\mathrm{d}\tau \\ &= L^{\uparrow}\left(\mathrm{d}P/\mathrm{d}\tau\right)L = L^{\uparrow}vLL^{\wedge}FL + \left(K\right)^{\uparrow} \end{aligned} \tag{6}$$

We see that all the L's can be canceled out here. This beautiful law is form covariant and it must come from a very elegant, if not pretty, law at the *quantum* level. This equation successfully predicts the path of a mass, with charge q, being pushed around by the external E and M fields in F. It does work. The moving charges make the F field and the F field pushes on the charges. It not only works, it all fits nicely into the H-P number system. I found this form for Newton's equation fairly recently. See my book for details [5]. It is *equivalent* to the old Lorentz force form: $q\boldsymbol{E} + qv\times\boldsymbol{B}$ in vector language, but we have bypassed all that machinery. Vectors do not need to be discussed to do the deepest physics of reality.

The non-physicist reader may not have the patience to fully digest all of this, but there is nothing hard to grasp here. It is mostly just *algebra* with a little partial differentiation thrown in, but never really used here. We have covered the known classical, relativistic world (except Einstein's beautiful curved-space gravity). We will bypass gravity here. It is in my book for those interested in seeing curved space in this language. Curved-space gravity was Einstein's really spectacular prediction, for he got the right equation at the classical approximation level for reality.

The World of Quantum Physics

We are now ready to *jump off into the much more complex world of quantum, relativistic physics,* again bypassing all of the non-relativistic, approximate stuff. We could examine an "alternative" equation to Maxwell's, in H-P, that looks similar but is not really so similar. In the absence of charge sources, $J = 0$, we see that Maxwell's equation becomes $\partial F = 0 \rightarrow L^{\uparrow}\partial LL^{\wedge} FL = 0$. Then what about the equation $L^{\uparrow}\partial LL^{\wedge}\psi = 0$? We are suggesting another kind of field here, ψ, where $\psi' = L^{\wedge}\psi$. (This is similar to the quantum Weyl equation.) Such classical fields seem natural here, in this H-P number system, but they are *not seen in our classical approximation* of the real world. Because there is only an L on the left of ψ, this equation naturally generalizes to $\partial\psi = K$, with some kind of "source" K satisfying $K' = L^{\uparrow}K$. Notice that here $\psi^{\uparrow\wedge} M \rightarrow (L^{\wedge}\psi)^{\uparrow\wedge} M = L^{\uparrow}[\psi^{\uparrow\wedge} M]$ for *any* M (hypercomplex) number in H-P; if invariant, $M' = M$. Thus $\partial\psi = \psi^{\uparrow\wedge} Mc$ is also form invariant. (Notice that the simpler form $P\psi = \psi Mc$ is only form covariant for the sub-group $L^{\wedge} = L^{\uparrow}$.) No such (classical) ψ field, with a mass parameter, has been found in our *classical* experience of the world. Our *quantum* world, with its successful ψ, does not reduce to this classical ψ field equation for a very significant reason, called the Pauli *exclusion principle.*

To *enter the quantum world,* we have to make guesses [9], [10], [11], [12], [13]. We directly measure things like the hydrogen atom's emitted photons and find a complicated pattern of spectral lines. These result *somehow* from the mathematical description of the electron interacting with the proton. These lines do not give us much help in trying to find the right laws for the electron in this bound situation, but they are an excellent *test* of our progress — they tell us when we have a successful theory. We must try to predict them, and we have learned how, to *ten-digit accuracy,* in the *QED* theory for hydrogen lines! This is an amazing fit of *our* mathematics to the *real* world. It shows that the concepts here are on the right track for our universe, but not why they work.

The right answer, to enter quantum, seems to be that we must first *double our number system,* before we even start to look for the quantum laws. We simply allow the *outside* coefficients to now become complex numbers. These c's have simple multiplication rules and we assume that $c(e_\mu) \equiv (e_\mu)c$ for any complex number $c \equiv a + Ib$, where $II \equiv -1$. Our new basis is now: $\{(e_\mu), (ie_\mu), I(e_\mu), I(ie_\mu)\}$, with 16 elements. The basis products now come in $16 \times 16 = 256$ pairs, but we can *still do all of these* in our heads!

We must now *extend the conjugations to these new I coefficients.* There is little doubt that we should define $I^{\uparrow} \equiv -I$. This is suggested by the matrix equivalent forms. However, matrices do not have the quaternion conjugation, so they give us no guidance here. About 30 years ago, I decided it is "obvious" that both conjugations are equally important so they should *both* treat I the same way. *I committed myself to* $I^{\wedge} \equiv -I$. In the last few months, I have begun to realize that this is not as natural as it may seem. We really should invent another conjugation that is like $(...)^\wedge$ except that $I^\Delta \equiv +I$. Both $(...)^\wedge$ and $(...)^\Delta$ are anti-automorphic. I am still working on the consequences of all this and will be reporting shortly, but so far it seems that the old conjugation is still more important. The new one is useful in combination with the old. If any $A^\wedge = A^\Delta$, then this A has no I parts.

This 16-basis system is large and complicated but nature uses it, so *we must also*. In fact, we must follow nature wherever required in the future. The $a + Ib$ coefficients might need extension to *quaternion coefficients,* $a + Ib + Jc + Kd$, with $\{I,J,K\}$ having the same multiplication table as $\{(-ie_1),(-ie_2),(-ie_3)\}$. We would expect $I^\wedge \equiv \pm I$ to be like $J^\wedge \equiv \pm J$ and $K^\wedge \equiv \pm K$, however defined. Now we have 32 basis elements and a $32 \times 32 = 1024$ entry multiplication table — it still can be done in our heads, but it is getting harder. This nice math pattern should be further explored to see if there is any physics in here. It works out well and is not too complicated.

I even came up with rules for multiplying *octonion coefficients* (see the book), with their eight basis elements: $\{(\sigma_0),(-i\sigma_k),(\sigma_0 j),(-i\sigma_k j)\}$, $k = 1,2,3$, so the 8×8 table can be done *in our heads* here as well. We recognize the quaternions as the first four parts. In some ways the octonions are a natural end of the coefficient generalization process. It was shown early in this century that no other systems have the "nice properties" of the quaternion and octonion systems, which we will not go into here. The number system now has $8 \times 16 = 128$ basis elements, as follows: $\{(\sigma_0)(e_\mu),(\sigma_0)(ie_\mu),(\sigma_0)(f_\mu),(\sigma_0)(if_\mu)$, $(-i\sigma_k)(e_\mu),...,(\sigma_0 j)(e_\mu),...,(-i\sigma_k j)(e_\mu),...$ And yes, we can still fill in the $128 \times 128 = 16{,}384$ element multiplication table *in our heads*! The e's commute with the σ's and that helps simplify the calculations considerably. This incredible number system may be the backbone of our physical reality. It is not pretty, but its use in the design may have been necessary for us to ever evolve here on the earth, to ponder all this. I have included this stretching of Dirac "to the limits" to show the reader that nature may not be simple at all. We just deal with the simple end of a complex tangle of equations for the world.

Back in the simple, 16-basis E system, $P^\uparrow \equiv +P$ has eight parts, not four. We only use four of these, *so far,* in our physics, but which four? (I wasted years on that problem.) The bottom line seems to be that, for quantum, nature uses

$$P = h\partial^\mu\left(e_\mu\right) = P^\uparrow \to L^\uparrow PL \tag{7}$$

The basic wave equations here are Maxwell, $PF = J$, and a totally new form of Dirac.

$$P\psi = \psi^{\uparrow\wedge}Mc + A\psi \tag{8}$$

This Dirac equation is form covariant if $A' = L^\uparrow AL$. This A is a new, external field affecting the ψ field as one of its sources. The A field is "derived" from the F field. (See my book for details.) This ψ field is also its own source, through the constant M, associated with the free quantum particle (cloud) described by this ψ field. Here, $\hbar$ and c are constants, one very small and the other very large. Some coupling constants have been left out above for clarity.

For several years I had hoped that this equation would turn out to be a new equation, but alas I have *recently proved that it is just the same as the equation Dirac found in 1928* for relativistic, quantum electrons. The similarity is *complicated* to show, because of the $\psi^{\uparrow\wedge}$ conjugations that are *needed* here for form covariance. There is nothing quite like that in ordinary matrix language!

The shocking new result here is that we have gotten all of the usual Dirac theory in only a 16-part number system! Dirac (and everyone since, as far as I know) had to go to a 32-part number system to get the Dirac equation to make sense in the old *matrix* language. Since we do not need that huge Dirac number system for all of the *Dirac physics,* then *what is full Dirac really good for?* Can we just throw full Dirac away? Maybe nature uses it at a deeper level — such as for the quark world (inside of protons) and for whatever is inside of electrons. Now that we realize there is more in all this Dirac machinery than we have really been using, we may find more exotic things there. The full Dirac algebra has the 32-basis set:$\{(e_\mu), (ie_\mu), I(e_\mu), I(ie_\mu); (f_\mu), (if_\mu), I(f_\mu), I(if_\mu)\}$. Now there are $32 \times 32 = 1024$ combinations, and my book shows how to still do *all of these* in our heads as well! We will not do much more with the Dirac algebra here. The E sub-algebra, consisting of the first 16 elements above, is *closed* and it probably is *big enough for the known physics.* That should be interesting to see proved or disproved, in the near future. This E system may simplify the quantum electrodynamics (QED) presentation in future textbooks.

We should next look at the natural covariance group in E, and later in D. We still find that $P = P^{\uparrow}$ and that $P^{\wedge} P$ is the correct inner product in E. Thus $LL^{\wedge} = 1(e_0)$ is still the natural group. The generator set now has the same old six generators, but there are two new ones: $I(e_0)$ and $I(ie_0)$. The first is a phase change, already seen in non-relativistic quantum theory. The second, however, turns a four-vector P into an eight-vector P, so *it is really radical*! We must throw it away or rethink the whole world of quarks. (Einstein's philosophical approach to covariance is clearly being left behind here. The math is now guiding us.) In *full* Dirac, the $LL^{\wedge} = 1(e_0)$ group has 16 parameters! One is again $I(e_0)$, and there are ten real and five (other) imaginary generators. We may have to find a way to cut these groups down to size, but *which* size? The ten real parameters suggest P might have five dimensions for full Dirac physics. The full 15 parameters go with a 15-part space-time. We do not really know what is going on down inside protons, or how many microscopic dimensions exist. We cannot go down there and look. Our elegant measuring tools are very crude, though ingeniously designed and huge in both size and cost.

If $P^{\wedge} P$ should NOT have any cross-terms in full Dirac, for some *physical* reason, then *P is limited to only five internal parts.* We may be able to justify throwing away any L parts that take P beyond these five parts, after the action $P' = L^{\uparrow}PL$. We *must find the correct form covariance group for nature* — assuming that such a covariance concept is still meaningful down at this 10^{-15} meter level, and below. I found another useful conjugation in *full* Dirac, $(...)^{\vee}$, which is also anti-automorphic. In E, $(...)^{\wedge} = (...)^{\vee}$, so they are closely related. If we guess that $L^{\wedge} \equiv L^{\vee}$, in the real world, then *the fifth part of P is also invariant, as M is invariant.* We then have *two candidates* for mass in the $P_{(5)}\psi = \psi^{\uparrow\wedge} Mc$ equation, but *only in full Dirac.* Remember that M here can be "anything" hypercomplex, without disturbing the covariance of the equation. We have to find a way to limit M from its maximum of 32 internal, invariant pieces! All kinds of wild things are possible here in full Dirac. The original equation, that he found in 1928, is essentially the case where $M = mI(f_0)$, and the *fifth part of P is possibly a "tachyonish" mass part that is being neglected.* Then Dirac's *original,* free particle equation looks like $P_{(4)}\psi \approx mI(f_0)c\psi$; whereas, Maxwell's equation in full Dirac now generalizes to $P_{(5)}F = 0$, with a fifth P part (tachyonish mass?). These results are probably just the tip of the "physics iceberg," here in full Dirac. Things are so much more restricted and thus simple if only the E system was really used in the design. That will have to be explored more in the future. Can QCD, the current theory of quarks, also be fitted into the smaller E hypercomplex number system?

The most amazing thing to me and probably to the reader is, "*How did our world come to be like this* — with all this *detailed* mathematical structure?" Could God change the mathematical rules a bit and still have a universe in which conscious humanoids like us evolve to contemplate it later? *Why go to full Dirac when the E sub-system is so much easier?* Is it possible that a universe designed around E may not lead to evolution of conscious beings? Maybe we can answer this *religious question* to some extent when we understand what quarks are really like, and *why* electrons are 1/1800th as heavy as protons. Everything in this paper could have been written in the 1930s! This is pre-second quantization, pre-QED theory. The perspective here is just different enough, from that we get from thinking with matrices, to have apparently been missed for about 70 years. I have spent 30 years developing this view "in a vacuum," largely because I was taught the old ways in graduate school before I started; it is so hard to break out. We must, as Feynman used to say, *look at things a new way.* He got me onto this life-consuming quest (my book is dedicated to his memory), and I know that he would revel in the E results now, were he still alive to see them. The other theorists who have dismissed this as trivial, even recently *(Journal of Mathematical Physics)*, are still missing a great opportunity. Schwinger also died recently but he had never responded to my earlier letters over the years. I have recently written to several theorists showing them briefly how Dirac fits in only *half* of the Dirac algebra, yet to be published. Perhaps they will now take this approach to covariance and field structure seriously, but I have had *no replies to date*. Whether they do take notice now, or continue to think this trivial, the real task lies ahead for the *next generation*. Maybe this article and the book will get a new *bandwagon* going, so we can still live to see *why electrons are lighter than protons,* and why muons are *200 times* heavier than electrons — instead of some *other* number. We know *nothing* really beyond chemistry, until we can explain these two numbers. This journal and its patient reviewer have been the *exception to the rule,* for my experience over the past few years. They are to be congratulated on their openness and courage, which have allowed you, the reader, the joy (or pain) of seeing the wonderful mathematical structure of the world, that *somehow* keeps us alive. It is certainly an incredible design.

Acknowledgments

S. Klein, W. Honig, K. Imaeda, and P. Lounesto have been encouraging this work over three decades. Recently, the author has benefitted from the encouragement (in some cases fleeting) of E. Marx, G. Dixon, J. Wheeler, L. Horwitz, F. Rohrlich, M. Jammer, S. Drell, L. Kauffman, A. Lakhtakia, W. Rodriguez, E. Recami, and S. Adler. The reviewer has been very helpful in making this more intelligible to the reader. (My book has a slow and methodical development of these ideas and their applications, aimed at budding theorists and curious outsiders, like George Carlin.)

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