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p-adic Information Spaces, Infinitely Small Probabilities and Anomalous Phenomena

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Abstract — A mathematical model of physical reality is proposed on the basis of the so-called p-adic numbers, providing p-adic probabilities. By this model, physical reality is information reality. Basic objects of this reality are transformators of information, basic processes are information processes, and the statistics are information statistics. The corresponding formalisms on information spaces are outlined. They can be used for the study of psychological and social phenomena as well as for anomalous phenomena as discussed in mind-matter research. The statistical behavior of information processes may strongly differ from statistical behavior of physical processes in real space (or space-time).

Keywords: p-adic numbers — information spaces — p-adic probability — anomalous phenomena

1. Introduction

Since Newton's time, the usual model of physical reality is based on a description of all physical processes by real numbers. To identify a physical quantity x with a real number, we have to assume the possibility to measure this quantity with infinite precision. Although this assumption is evidently idealistic, there is a great number of physical phenomena which are well described this way. However, the "real" model of physical reality does not describe all empirical phenomena. For example, there is no place for many psychological, social or anomalous phenomena in this model.

We propose a new mathematical model of physical reality based on a system of p-adic numbers (see Section 3 and [1] for this number system). By our model physical reality is not a reality of matter alone, which is located in continuous real space (or space-time, respectively), but it is *information reality*. Basic elements of this reality are the so-called *transformators of information*; basic processes are processes of an exchange of information. By our model information processes have no rigid relation to processes in real space (or space-time). There may be a large difference between distances in real space and information space (or space-time). This might imply the exchange of information between transformators located at large distances in real space. This exchange might be realized without an exchange of physical (real) energy; only the so-called *information energy* is required. The difference between real time and *information time* might imply an information exchange with the past or the future.

The statistical behavior of information processes may strongly differ from the statistical behavior of real physical processes (*i.e.*, processes in real space). To describe this new statistics we use a p-adic probability theory which has been developed in [2]. In particular, the p-adic framework allows us to study "random" sequences without the property of statistical stabilization (in the ordinary sense), *i.e.*, the law of large numbers is violated for these sequences, see [2] for details. Another interesting feature is the p-adic realization of infinitely small probabilities. In our formalism events which have zero probability in the ordinary probabilistic formalism may not be negligible. They are labeled by *negative probabilities* and play an important role in probabilistic considerations.

2. "Real" Reality and Information Reality

To realize a measurement of a physical quantity x, first we have to fix a unit $l = 1$ of a measurement. We assume that there exists a natural number n such that

$$(n - 1)l \leq x < nl. \tag{1}$$

This assumption is a mathematical postulate, the *Archimedean axiom.* By (1) we restrict our considerations to physical phenomena which can be described on the basis of the Archimedean mathematical model.

Now we consider the next step of the measurement process. If $y_1 = (n-1)l \neq x$ then we have to measure the quantity $x_1 = x - y_1$ by using a unit smaller than l. Typically we fix a natural number $m > 1$ (the scale of the measurement) and choose the new unit $l_1 = l/m$. Then we apply the Archimedean axiom (1) to the quantities x_1 and l_1 and obtain a natural number β_1 $(\beta_1 = 1, \ldots, m)$: $(\beta_1 - 1)l_1 \leq x_1 < \beta_1 l_1$. This procedure can be continued. If $y_2 = (\beta_1 - 1)l_1 \neq x_1$ then we can use the new unit of measurement $l_2 = l_1/m$ to measure the quantity $x_2 = x_1 - y_2$, and so on. We remark that

$$\begin{aligned} x &= (n-1) + x_1 = (n-1) + \frac{\alpha_{-1}}{m} + x_2 \\ &= n - 1 + \frac{\alpha_{-1}}{m} + \cdots + \frac{\alpha_{-n}}{m^n} + x_{n+1}, \end{aligned} \tag{2}$$

where $\alpha_{-k} = \beta_k - 1 = 0, 1, \ldots, m-1$. To obtain the real numbers model for physical reality, we assume that the above process of measurements of every physical quantity x can be continued by an infinite number of steps. We call this postulate the postulate of an infinite precision of measurements or the *Newtonian axiom.* By this axiom any physical quantity x can be identified with a real number:

$$x = \cdots + \frac{\alpha_{-n}}{m^n} + \cdots + \frac{\alpha_{-1}}{m} + \alpha_0 + \alpha_1 m + \cdots + \alpha_k m^k$$

$$= \alpha_k \cdots \alpha_0, \alpha_{-1} \cdots \alpha_{-n} \cdots, \tag{3}$$

where $\alpha_{\pm j} = 0, 1, \ldots, m - 1$ (here the number $(n - 1)$, see (2), is also expanded with respect to powers of m).

Both the Archimedean and Newtonian axioms are natural for the description of an extended class of physical phenomena. This class of physical phenomena was intensively studied in the last 200–250 years. The basis of this reality is Newton's space which is continuous, infinitely divisible and infinitely deep. All physical objects are located in this space and their location can be determined (at least in principle) with infinite precision. The idea of space is the primary idea, and it is impossible to imagine any physical phenomenon which is not located in space — a conception which is very much in a Kantian spirit.

It might seem strange that Kants's idealistic philosophy is so closely related to modern physics. However, this is not quite so strange, because the Archimedean-Newtonian mathematical model is idealistic in a similar way. The postulate about an infinite precision of measurements is non-empirical. Physical quantities cannot be measured (determined) with infinite precision. Any physical phenomenon has a finite limit of its description, see [3] for the details. Therefore we cannot consider Archimedean-Newtonian reality as the unique model of physical reality. Physical reality should not be identified with this "real" reality.

It is natural to develop other models of physical reality which are not based on the Archimedean and Newtonian axioms. The quantum formalism is one successful attempt to provide such a new model of physical reality. The Archimedean and Newtonian axioms cannot be applied to quantum observables. However, quantum theory uses the mathematical basis of real numbers.

Another model of physical reality has been developed in [3] (see also [5]). It is based on two postulates: (1) Non-Archimedean postulate: there exist physical quantities x which cannot be measured (even approximately) on the basis of the fixed unit of measurement $l = 1$. Such quantities are interpreted as infinitely large. (2) Non-Newtonian postulate: if a quantity x can be measured on the basis of the unit of measurement $l = 1$, then this measurement has a finite precision, *i.e.*, the process (2) has to stop after a finite number of steps.

Such quantities can be naturally described by the so-called m-adic numbers (see Section 3 and [1] for m-adic numbers). In this model we do not consider physical objects as structures located in a continuous space (or space-time, respectively). Physical objects have an unsharp m-adic description with a finite precision. At the same time, this precision is the boundary of existence of a physical object. If we want to create a new model we must not follow the ideas about "physical space-time." The properties of this "physical space-time" are mainly the properties of the Archimedean-Newtonian model.

In this paper we develop a model of physical reality based on the idea that physical processes and phenomena are not realized in the Archimedean-Newtonian (real) “physical space,” but they are realized in information spaces. Thus physical objects are information objects and physical processes are information processes. The “ordinary physical objects” give a particular class of information objects. These are information objects with stable or slowly changing information states. Our model of information reality is based on the system of m-adic numbers. For purely mathematical reasons, we use the system of p-adic numbers where $m = p > 1$ is a prime number. An essential part of the physical apparatus for this information model has been developed in the framework of p-adic classical and quantum physics [2, 3, 5–15] (p-adic strings, quantum mechanics, field theory).

3. Systems of p-adic Numbers

The field of real numbers **R** is constructed as the completion of the field of rational numbers **Q** with respect to the metric $\rho(x, y) = |x - y|$, where $|\cdot|$ is the usual valuation given by the absolute value. The fields of p-adic numbers $\mathbf{Q}_p$ are constructed in a corresponding way, but using other valuations. For a prime number p, the p-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers. Every natural number n can be represented as the product of prime numbers, $n = 2^{r_2}3^{r_3}\cdots p^{r_p}\cdots$, and we define $|n|_p = p^{-r_p}$, writing $|0|_p = 0$ and $|-n|_p = |n|_p$. We then extend the definition of the p-adic valuation $|\cdot|_p$ to all rational numbers by setting $|n/m|_p = |n|_p/|m|_p$ for $m \neq 0$. The completion of **Q** with respect to the metric $\rho_p(x, y) = |x - y|_p$ is the locally compact field of p-adic numbers $\mathbf{Q}_p$. The number fields **R** and $\mathbf{Q}_p$ are unique insofar as, by Ostrovsky’s theorem (see [1]), $|\cdot|$ and $|\cdot|_p$ are the only possible valuations on **Q**, but they have quite distinctive properties. Unlike the absolute value distance $|\cdot|$, the p-adic valuation satisfies the strong triangle inequality $|x + y|_p \leq \max[|x|_p, |y|_p]$, $x, y \in \mathbf{Q}_p$.

Write $U_r(a) = \{x \in \mathbf{Q}_p : |x - a|_p \leq r\}$ and $U_r^-(a) = \{x \in \mathbf{Q}_p : |x - a|_p < r\}$, where $r = p^n$ and $n = 0, \pm 1, \pm 2, \ldots$. These are the “closed” and “open” balls in $\mathbf{Q}_p$ while the sets $S_r(a) = \{x \in K : |x - a|_p = r\}$ are the spheres in $\mathbf{Q}_p$ of such radii r. These sets (balls and spheres) have a somewhat strange topological structure from the viewpoint of our usual Euclidean intuition: they are both open and closed at the same time, and as such they are called *clopen* sets. Another interesting property of p-adic balls is that two balls have non-empty intersection if and only if one of them is contained in the other. Also, we note that any point of a p-adic ball can be chosen as its center, so that such a ball is not uniquely characterized by its center and radius. Finally, any p-adic ball $U_r(0)$ is an additive subgroup of $\mathbf{Q}_p$, while the ball $U_1(0)$ is also a ring, which is called the *ring of p-adic integers*, denoted by $\mathbf{Z}_p$.

Any $x \in \mathbf{Q}_p$ has a unique canonical expansion (which converges in the $|\cdot|_p$-norm) of the form $x = a_{-n}/p^n + \cdots a_0 + \cdots + a_k p^k + \cdots$ where the $a_j \in \{0, 1, \ldots, p-1\}$ are the “digits” of the p-adic expansion. The elements

$x \in \mathbf{Z}_p$ have the expansion $x = a_0 + \cdots + a_k p^k + \cdots$ and can thus be identified with the sequences of digits $x = a_0 \cdots a_k \cdots$.

We remark that the set of natural numbers $\mathbf{N}$ is contained in $\mathbf{Z}_p$ and every natural number has a finite number of non-zero digits in its expansion. We interpret elements $x \in \mathbf{Z}_p$ which have an infinite number of non-zero digits in their expansion as *infinitely large natural numbers*.

If, instead of a prime number p, we start with an arbitrary natural number $m > 1$ we construct the system of the so-called m-adic numbers $\mathbf{Q}_m$ by completing $\mathbf{Q}$ with respect to the m-adic metric $\rho_m(x, y) = |x - y|_m$ which is defined in a way similar to the above. However, this system is in general not a field as there may exist divisors of zero.

4. Dynamics on Information Spaces and Anomalous Phenomena

The rings of p-adic integers $\mathbf{Z}_p$ can be used as mathematical models for information spaces. Each element $x = \sum_{j=0}^{\infty} \alpha_j p^j$ can be identified with a sequence

$$x = \alpha_0 \alpha_1 \cdots \alpha_N \cdots, \qquad \alpha_j = 0, 1, \ldots, p - 1. \tag{4}$$

Such sequences are interpreted as coding sequences (in the alphabet $A_p = \{0, 1, \ldots, p - 1\}$ with p letters) for some amount of information. The p-adic metric $\rho_p(x, y) = |x - y|_p$ on $\mathbf{Z}_p$ is a measure for the nearness of information sequences: two sequences x and y are close if they have a sufficiently long common initial segment. Such nearness between information sequences is nothing but the associative property of information. In fact, we can identify two information sequences which have the same initial segment of large length. We choose the space $X = \mathbf{Z}_p$ for the description of information. This space is said to be an *information space*, briefly I-space.

We do not assume that X describes the whole information in the universe. In our approach, there is no absolute physical space (which is typically identified with the real space). There is also no absolute information space. Different information phenomena can be described by different mathematical models for I-spaces. The p-adic model for I-spaces is the simplest model from a mathematical point of view. In principle, we may use more complicated mathematical structures (for example, the rings $\mathbf{Z}_m$).

Objects which "live" in I-spaces are said to be *transformators of information* (I-transformators). Each I-transformator τ has an internal I-state $t \in T = \mathbf{Z}_p$ which is called I-time.[1] At each instant $t \in T$ of I-time τ has some external I-state $q(t) \in X$ which describes the position of τ in the configuration I-space X. The "life"-trajectory of τ can be identified with the trajectory $q(t)$ in X.

[1] I-time can be related to psychological time in models which are used for psychological or anomalous phenomena.

We can now develop an analog to Hamiltonian dynamics on the I-spaces. As usual, we introduce the quantity $p(t) = \dot{q}(t)$ $(= (d/dt)q(t))$ which is the information analog of momentum. However, here we prefer to use a psychological terminology. The quantity $p(t)$ is said to be a *motivation* (for changing the I-state $q(t)$).

The space $\mathbf{Z}_p \times \mathbf{Z}_p$ of points $z = (q, p)$ where q is the I-state and p is the motivation is said to be a phase I-space. As in the ordinary Hamiltonian formalism, we assume that there exists a function $H(q, p)$ (I-Hamiltonian) on the phase I-space which determines the motion of τ in the phase I-space:

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t)), \qquad q(t_0) = q_0, \tag{5}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t)), \qquad p(t_0) = p_0. \tag{6}$$

The I-Hamiltonian $H(p, q)$ has the meaning of an I-energy. In principle, I-energy is not related to the usual physical energy.

The simplest I-Hamiltonian $H_{\mathrm{f}}(p) = \alpha p^2$, $\alpha \in Z_p$ describes the motion of a free I-transformation τ, *i.e.*, an I-transformator which uses only self-motivations for changing its I-state $q(t)$. Solving the system of the Hamiltonian equations we obtain: $p(t) = p_0$, $q(t) = q_0 + 2\alpha p_0(t - t_0)$.[2] The motivation $p(t)$ is the constant of this motion. Thus the free I-transformator "does not like" to change its motivation p_0 in the process of the motion in the I-space. If we change coordinates, $q' = (q - q_0)/k$, $k = 2\alpha p_0$, then we see that the dynamics of the free I-transformator coincides with the dynamics of its I-time.

In the general case the I-energy is the sum of the I-energy of the motivations $H_{\mathrm{f}} = \alpha p^2$ (which is an analog of the kinetic energy) and potential I-energy $V(q)$:

$$H(q, p) = \alpha p^2 + V(q).$$

The potential $V(q)$ is determined by *fields of information*.

In the Hamiltonian framework we can consider interactions between different I-transformators, $\tau_1, \ldots, \tau_N$. These I-transformators have the I-times $t_1, \ldots, t_N$ and I-states $q_1(t_1), \ldots, q_N(t_N)$. By our model we can describe interactions between these I-transformators only if there is a possibility to choose the same I-time t for all of them. In this case we can consider the evolution of the system of the I-transformators $\tau_1, \ldots, \tau_N$ as a trajectory in the I-space $\mathbf{Z}_p^N = \mathbf{Z}_p \times \cdots \times \mathbf{Z}_p$, $q(t) = (q_1(t), \ldots, q_N(t))$. We think that this condition of consistency for I-times of interacting I-transformators plays the crucial role in psychological experiments. We cannot obtain sensible observations for interactions between arbitrary individuals. There must be a process of learning

[2] In fact, this simplest I-system is not so simple from a mathematical viewpoint. There exist other solutions which are non-analytical (but smooth), see [1]. These solutions may also have an interesting I-interpretation. But we shall not discuss this problem here.

for the group $\tau_1, \ldots, \tau_N$ which reduces individual I-times $t_1, \ldots, t_N$ to the unique I-time t.

Thus, let us consider a group $\tau_1, \ldots, \tau_N$ of I-transformators with the internal I-time t. The dynamics of I-states and motivations is determined by the I-energy; $H(q, p)$, $q \in \mathbf{Z}_p^N$, $p \in \mathbf{Z}_p^N$. It is natural to assume that

$$H(q, p) = \sum_{j=1}^{N} \alpha_j p_j^2 + V(q_1, \ldots, q_N), \qquad \alpha_j \in \mathbf{Z}_p.$$

Here $H_f(p) = \sum_{j=1}^{N} \alpha_j p_j^2$ is the total energy of motivations for the group $\tau_1, \ldots, \tau_N$ and $V(q)$ is the potential energy. It is natural to choose $V(q) = \sum_{i \neq j} \Phi(q_i - q_j)$, where $\Phi(s)$, $s \in \mathbf{Z}_p$, is the potential of the interaction between I-transformators.

As usual, to find a trajectory on the phase I-space $\mathbf{Z}_p^N \times \mathbf{Z}_p^N$, we need to solve the system of Hamiltonian equations:

$$q_j = \frac{\partial H}{\partial p_j}, \quad p_j = -\frac{\partial H}{\partial q_j}, \quad q_j(t_0) = q_0, \quad p_j(t_0) = p_0. \tag{7}$$

(See [2] for such equations).

5. Consequences for Psychology and Mind-Matter Research

5.1. Energy and Information

In our model the transmission of information is determined by the I-energy which is the sum of the I-energy of motivations and potential I-energy. In principle, this process does not need to involve physical energy. Therefore, there might be transmissions of information which could not be reduced to transmissions of physical energy. In this case we cannot measure physical interactions between two I-transformators, τ_1 and τ_2. In particular, τ_1 and τ_2 can be individuals participating in psychological experiments or in experiments dealing with anomalous phenomena.

5.2. Distance and Information

All information processes are realized in I-space rather than in physical space. Therefore the (real) physical distance between I-transformators does not play the crucial role in processes of I-transmission and I-interactions.

5.3. Time and Information

The dynamics of I-states cannot be considered as dynamics with respect to (real) physical time. The internal (psychological) time is strongly involved in I-processes. In particular, the (real) order structure for (real) physical instants of time does not determine I-processes. Moreover, the (real) physical distance between instants of physical time t_{phys} can strongly differ from the distance (in

I-space) between the corresponding instants of I-time t. For example, let us assume that we can construct a correspondence between physical time and I-time, $t_{\text{phys}} = g(t)$, $t = \phi(t_{\text{phys}})$. Then the real distance $\rho_{\mathbf{R}}(t_{\text{phys}} - t'_{\text{phys}})$ may be rather large, while $\rho_p(t - t')$ may be very small. Thus, $\rho_p(q(t) - q(t'))$ may also be very small for every continuous trajectory $q(t)$. If $t_{\text{phys}} < t'_{\text{phys}}$ then we can consider such a situation as the act of predicting the I-state, $a' = q(t')$. If $t_{\text{phys}} > t'_{\text{phys}}$ then we can consider such a situation as the act of recalling. There is a symmetry between processes of predicting and recalling of information. Of course, both these processes are unsharp processes, *i.e.*, it is impossible to predict or recall the information a' exactly. It is possible to predict or recall only some information $a = q(t)$ which does not differ strongly from a' (with respect to the distance in I-space).

5.4. Motivation

A motion in I-space depends not only on the initial I-state q_0, but also on the initial motivation q_0. Moreover, the Hamiltonian structure of the equations of motion implies that the motivations $p(t)$ play an important role in the process of evolution. Thus the I-dynamics is, in fact, a dynamics in the phase I-space.

5.5. Consistency for Times

An I-interaction between I-transformators is possible only if these I-transformators have consistent I-times. Therefore every psychological experiment has to contain an element of “learning” for I-transformators participating in the experiment. This learning does not need to be a physical interaction. It can be any exchange of information between individuals (or a study of information about some individual).

5.6. Future and Past

The consistency condition for I-times does not imply such a condition for (real) physical times, because different I-transformators can have different correspondence laws for I-times and (real) physical time. For example, let us consider two I-transformators, τ_1 and τ_2, satisfying the consistency condition for I-times, *i.e.*, $t_1 = t_2 = t$. We assume that it is possible to transform I-times of τ_1 and τ_2 to (real) physical times $t_{1,\text{phys}} = g_1(t_1)$ and $t_{2,\text{phys}} = g_2(t_2)$. Let us also assume that τ_1 and τ_2 interact by the I-potential $V(q_1 - q_2)$, *i.e.*, at the instant t of I-time the potential I-energy of this interaction equals $V(q_1(t) - q_2(t))$. If $t_{1,\text{phys}} = g_1(t_1) \neq t_{2,\text{phys}} = g_2(t_2)$ then such an interaction is nothing but an interaction with the future or the past.

6. Quantum Mechanics on Information Spaces

It is quite natural to quantize classical mechanics on information spaces over $\mathbf{Z}_p$. As usual in quantum theory, we can assume that a value λ of an I-quantity A can be measured in the state ϕ with some probability $\mathbf{P}_\phi(A = \lambda)$. This is nothing but the application of the *statistical interpretation of quantum mechanics* (see, for example, [16]) to information theory. By this interpretation any measurement process has two steps: (1) a preparation procedure $\mathcal{E}$; (2) a measurement of a quantity B in the states ϕ which were prepared by $\mathcal{E}$.

Let us consider these steps in the information framework. By $\mathcal{E}$ we have to select a statistical ensemble ϕ of I-transformators on the basis of some I-characteristics. Typically in quantum physics a preparation procedure $\mathcal{E}$ is realized as a filter based on some physical quantity A, *i.e.*, we select elements which satisfy the condition $A = \mu$ where μ is one of the values of A. We can do the same in quantum I-theory. An I-quantity A is chosen as a filter, *i.e.*, I-transformators for the statistical ensemble ϕ are selected by the condition $A = \mu$ where $\mu \in \mathbf{Z}_p$ is some information. For example, we can choose $A = p$, the motivation, and select a statistical ensemble $\phi = \phi\ (p = \mu)$ of I-transformators which have the same motivation $\mu \in \mathbf{Z}_p$. Then we realize the second step of a measurement process and measure some I-quantity B in the state $\phi_{(p=\mu)}$. For example, we can measure an I-state q of I-transformators belonging to the statistical ensemble described by $\phi_{(p=\mu)}$. We shall obtain a probability distribution $\mathbf{P}(q = \lambda | p = \mu)$, $\lambda, \mu \in \mathbf{Z}_p$ (a probability that the I-transformator is in I-state $q = \lambda$ under the condition that it has the motivation $p = \mu$). Considering the I-energy E of I-transformators, we obtain a probability distribution $\mathbf{P}(E = \lambda | p = \mu)$, $\lambda, \mu \in \mathbf{Z}_p$.[3]

On the other hand, we can prepare a statistical ensemble $\phi_{(q=\mu)}$ by fixing some information $\mu \in \mathbf{Z}_p$ and selecting all I-transformators which are in I-state $q = \mu$. Then we can measure motivations of these I-transformators and obtain a probability distribution $\mathbf{P}(p = \lambda | q = \mu)$. In fact, a mathematical model for a quantum I-formalism has been constructed earlier: a quantum mechanics with p-adic valued functions, see [2, 3, 5, 14, 15]. We now propose a new interpretation (the *information interpretation*) for this model.

Roughly speaking (for details, see [2, 3, 5, 14, 15]), the space of quantum states is realized as a p-adic Hilbert space $\mathcal{K}$ (see [2] for the theory of such spaces). This is a $\mathbf{Q_p}$-linear space which is a Banach space (with the norm $\|\cdot\|$) and on which a symmetric bilinear form $(\cdot, \cdot) : \mathcal{K} \times \mathcal{K} \to \mathbf{Q_p}$ is defined. This form is called an inner product on $\mathcal{K}$. It is assumed that the norm and the inner product are connected by the Cauchy-Bunaykovski-Schwarz inequality:

$$|(x, y)|_p \leq \|x\|\|y\|, \ x, y \in \mathcal{K}$$

It is possible to use more general spaces over different extensions of $\mathbf{Q_p}$ (analogs of complex Hilbert spaces).

By definition, a quantum I-state ϕ is an element of $\mathcal{K}$ such that $(\phi, \phi) = 1$; a quantum I-quantity A is a symmetric bounded operator $A : \mathcal{K} \to \mathcal{K}$, *i.e.*,

[3] We do not discuss concrete measurement procedures for I-quantities. In particular, at the moment it is not clear how I-energy can be measured. It seems natural to use an analogy with usual quantum theory here. The I-energy can be measured in the process of interactions between I-transformators or interactions of I-transformators and I-fields.

$(Ax, y) = (x, Ay)$, $x, y \in \mathcal{K}$.[4] We discuss a statistical interpretation of quantum states in the case of a discrete spectrum of A.

Let $\{\lambda_1, \ldots, \lambda_n, \ldots\}$, $\lambda_j \in \mathbf{Z}_p$ be eigenvalues of A, $A\phi_n = \lambda_n \phi_n$, $\phi_n \in \mathcal{K}$, $(\phi_n, \phi_n) = 1$. The eigenstates ϕ_n of A are considered as pure quantum I-states for A, *i.e.*, if the system of I-transformators is described by the state ϕ_n then the I-quantity A has the value $\lambda_n \in \mathbf{Z}_p$ with probability 1. Let us consider a mixed state

$$\phi = \sum_{n=1}^{\infty} q_n \phi_n, \qquad q_n \in \mathbf{Q_p}, \tag{8}$$

where $(\phi, \phi) = \sum_{n=1}^{\infty} q_n^2 = 1$.[5] If we realize a measurement of the I-quantity A for I-transformators belonging to the statistical ensemble described by ϕ, then we obtain the value λ_n with probability $P(A = \lambda_n | \phi) = q_n^2$.

The main problem (or advantage?) of this quantum model is that these probabilities belong to the field of p-adic numbers $\mathbf{Q_p}$. The simplest way to eliminate this problem is to consider only finite mixtures (8) for which $q_n \in \mathbf{Q}$ (the field of rational numbers $\mathbf{Q}$ is a subfield of $\mathbf{Q_p}$). In this case the quantities $\mathbf{P}(A = \lambda_n | \phi) = q_n^2$ can be interpreted as usual probabilities (for example, in the framework of Kolmogorov's theory [17]). Therefore we may assume that quantum I-states ϕ can be prepared which have the standard statistical interpretation: when the number N of experiments tends to infinity, the frequency $\nu_N(A = \lambda_n | \phi)$ of an observation of the information $\lambda_n \in \mathbf{Z}_p$ tends to the probability q_n^2.

However, we can approach this problem from a more general viewpoint. By the frequency model for p-adic probabilities, non-Kolmogorov probabilities are defined as limits of relative frequencies ν_N but with respect to a p-adic topology [2]. The relative frequencies ν_N belong to $\mathbf{Q}$ and they can be considered not only as elements of $\mathbf{R}$, $\mathbf{R} \supset \mathbf{Q}$, but also as elements of $\mathbf{Q}_p$, $\mathbf{Q}_p \supset \mathbf{Q}$. This probability model is a natural generalization of the frequency probability model by von Mises [18].

Using the p-adic frequency probability model for the statistical interpretation of quantum I-states we may assume that there exist I-states ϕ (ensembles of I-transformators) such that the relative frequencies $\nu_N(A = \lambda_n | \phi)$ have no limit in $\mathbf{R}$, *i.e.*, we cannot apply the standard law of the large numbers in this situation (compare [19]). Hence if we realize measurements of I-quantity A for such a quantum I-state and try to study the observed data by using standard statistical methods based on real analysis, then we will not obtain any definite result. There will be random fluctuations of relative frequencies, see [2].

Such behavior might be related to psychological experiments and anom-

[4] In p-adic models we do not need to consider unbounded operators, because all quantum quantities can be realized by bounded operators, see [3, 14, 15]

[5] As in the usual theory of Hilbert spaces, eigenvectors corresponding to different eigenvalues of a symmetric operator are orthogonal.

alous phenomena. Unfortunately, this case cannot be easily studied though by using p-adic analysis. In principle, we do not know what prime number p gives the basis of the concrete I-model ($p = 2$? or $p = 1997$? ...). Moreover, the I-model can be described by some other number system (for example, m-adic numbers where m is not a prime).[6] The direct (but not simple) way to determine a suitable p is to study fluctuating statistical data for different values of $p = 2, 3, \ldots$ and try to find the statistical stabilization with respect to one of the metrics ρ_p. Anyway, our theory has an important consequence for scientists doing experiments with statistical I-data: the absence of the statistical stabilization (random fluctuation) does not imply the absence of an I-phenomenon. Lacking statistical stabilization may have the meaning that an I-phenomenon cannot be described by the standard Kolmogorov probability model.

Another interesting application of p-adic probability theory concerns statistical samples [2] in which the frequencies $\nu_N \to 0$ in the standard real topology, but $\nu_N \to \alpha \neq 0$ in $\mathbf{Q_p}$. In this case the usual (Mises) probability $\mathbf{P}_{\text{Mis}}(A = \lambda|\phi) = 0$. This implies that we have to consider the event $\{A = \lambda|\phi\}$ (an observation of the information λ) as a non-physical event. However, from the point of view of p-adic probability theory, it is a physical event in the sense of I-physics. In the next sections we discuss the problem of infinitely small probabilities more carefully on the basis of the proportional definition of probability.

7. *p*-adic Proportional Probabilities

Our interpretation of p-adic numbers

$$N = l_0 + l_1 p + \cdots + l_s p^s + \cdots, \tag{9}$$

where $l_s = 0, 1, \ldots, p-1$, with an infinite number of non-zero digits l_s as infinitely large numbers gives the possibility of considering numerous actual infinities. Therefore we can study proportional probabilities on populations of an infinite volume. We shall study a population $\Omega = \Omega_N$ which has volume N, where N is the p-adic integer (9). If N is finite then Ω is the ordinary finite population. If N is infinite then Ω essentially has a p-adic structure. Consider a sequence of populations M_s with volumes $l_s p^s$, $s = 0, 1, \ldots$. For $\Omega = \cup_{s=0}^{\infty} M_s$, we have $|\Omega| = N$.

We may imagine a population Ω as the population of a tower $T = T_\Omega$ which has an infinite number of floors where the population of the sth floor is M_s. Then, $T_k = \cup_{s=0}^{k} M_s$ is the population of the first $k+1$ floors. For $A \subset \Omega$ and the limit

$$n(A) = \lim_{k\to\infty} n_k(A), \quad \text{where } n_k(A) = |A \cap T_k|, \tag{10}$$

[6] Of course, there may be some theoretical reasons for the choice of p, see [2].

we define the probability of A by the standard proportional relation:

$$\mathbf{P}(A) \equiv \mathbf{P}_\Omega(A) = \frac{n(A)}{N}. \tag{11}$$

We denote the family of all $A \subset \Omega$, for which (11) exists, by $\mathcal{F} = \mathcal{F}_\Omega$. The sets $A \in \mathcal{F}$ are said to be events. Later we shall study some properties of the family of events. First we consider the set algebra F which consists of all finite subsets and their complements, $F \subset \mathcal{F}$. We have the following general results for p-adic proportional probabilities:

Proposition 1. *Let $A_1, A_2 \in \mathcal{F}$ and $A_1 \cap A_2 = \emptyset$. Then $A_1 \cup A_2 \in \mathcal{F}$ and* $\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2)$.

Proposition 2. *Let $A_1, A_2 \in \mathcal{F}$. The following conditions are equivalent:*

$$1) A_1 \cup A_2 \in \mathcal{F}; \qquad 2) A_1 \cap A_2 \in \mathcal{F};$$
$$3) A_1 \setminus A_2 \in \mathcal{F}; \qquad 4) A_2 \setminus A_1 \in \mathcal{F}.$$

There are standard formulas:

$$\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2) - \mathbf{P}(A_1 \cap A_2); \tag{12}$$
$$\mathbf{P}(A_1 \setminus A_2) = \mathbf{P}(A_1) - \mathbf{P}(A_1 \cap A_2). \tag{13}$$

Definition. *The system $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P})$ is called a p-adic proportional probability space for the population of the volume $|\Omega| = N$.*

In fact, any proportional probability space $\mathcal{P}$ can be approximated by proportional probability spaces $\mathcal{P}_k$ with populations of finite volumes. Set

$$N_k = l_0 + l_1 p + \cdots + l_k p^k$$

for N which has the expansion (9). Let l_s be the first non-zero digit in (9). Consider populations Ω_{N_k}, $k = s, s+1, \ldots,$ and proportional probability spaces $\mathcal{P}_{N_k} = (\Omega_{N_k}, \mathcal{F}_{N_k}, \mathbf{P}_{N_k})$. Then $\mathcal{F}_{N_k}$ coincides with the algebra F_{N_k} of all subsets of Ω_{N_k} and

$$\mathbf{P}_{N_k}(A) = \frac{|A|}{|\Omega_{N_k}|}, \qquad A \in F_{N_k}. \tag{14}$$

If we identify Ω_{N_k} with the population of the first $k+1$ floors of the tower T_Ω, we have

Proposition 3. *Let $A \in \mathcal{F}_\Omega$. Then*

$$\mathbf{P}(A) = \lim_{k\to\infty} \mathbf{P}_{N_k}(A \cap \Omega_{N_k}). \tag{15}$$

8. Rules for Working with p-adic Probabilities

One of the main tools of the ordinary theory of probability is based on the order structure on the field of real numbers $\mathbf{R}$. It provides the possibility of comparing probabilities of different events. Events E with probabilities $\mathbf{P}(E) \ll 1$ are considered as negligible and events E with probabilities $\mathbf{P}(E) \approx 1$ are considered as practically certain. However, the use of these relations in concrete applications is essentially based on our (real) probability intuition.

What is a large probability? What is a small probability? Ordinary probability intuition is based on centuries of human experience rather than on exact mathematical theory. If we want to work with p-adic probabilities we have to develop some kind of a p-adic probability intuition. Trying to achieve such an intuition, we meet a mathematical problem which prevents us from generalizing the real scheme directly. This is the absence of an order structure on $\mathbf{Q}_p$.

However, there is a partial order structure on the ring of p-adic integers. Let $x = x_0 x_1 \ldots x_n \ldots$ and $y = y_0 y_1 \ldots y_n \ldots$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$. We set $x < y$ if there exists n such that $x_n < y_n$ and $x_k \leq y_k$ for all $k > n$. This partial order structure on $\mathbf{Z}_p$ is the natural extension of the standard order structure on the set of natural numbers $\mathbf{N}$. It is easy to see that $x < y$ for any $x \in \mathbf{N}$ and $y \in \mathbf{Z}_p \setminus \mathbf{N}$, *i.e.*, any finite natural number is less than any infinite number. But we cannot compare any two infinite numbers.

It is important to notice that there exists a maximal number $N_{\max} \in \mathbf{Z}_p$. This is easy to see: $N_{\max} = -1 = (p-1) + (p-1)p + \cdots + (p-1)p^n + \cdots$. Therefore the population Ω_{-1} is the largest population which can be considered in the p-adic framework. We can, in fact, restrict our considerations to the case of the maximal population Ω_{-1}. Let us study this case, $\Omega \equiv \Omega_{-1}$.

Let $A, B \in \mathcal{F}$. By definition $\mathbf{P}(A) < \mathbf{P}(B)$ if $n(A) < n(B)$. Further we study the properties of probabilities: (1) as we have only a partial order structure we cannot compare probabilities of two arbitrary events A and B; (2) as $x \leq -1$ for any $x \in \mathbf{Z}_p$, we have $\mathbf{P}(A) \leq 1 = \mathbf{P}(\Omega)$ for any $A \in \mathcal{F}$; (3) as $0 \leq x$ for any $x \in \mathbf{Z}_p$, we have $0 \leq \mathbf{P}(A)$ for any $A \in \mathcal{F}$. To illustrate further properties of p-adics we shall refer to the usual real order structure and denote increases or decreases of $\mathbf{P}$ as based on the real numbers by "r-increase" or "r-decrease." We shall use the symbols $[a, b], \ldots, (a, b)$ for corresponding intervals of the real line.

Set $F^{\mathrm{f}} = \{A \in \mathcal{F} : n(A) \in \mathbf{N}\}$ (in particular, F^{f} contains all finite subsets of Ω). Because the partial order structure on N coincides with the standard order structure, we have the standard order relation for $n(A), n(B)$, where $A, B \in F^{\mathrm{f}}$. But $\mathbf{P} : F^{\mathrm{f}} \to (-\infty, 0) \cap \mathbf{Z}$ and $\mathbf{P}(A)$ is increasing if $\mathbf{P}(A)$ is r-decreasing to $-\infty$. Therefore, for example, probabilities $\mathbf{P}(A) = -1$ or -3 are rather small with respect to probabilities $\mathbf{P}(B) = -100$ or -300.

Set $\bar{F}^{\mathrm{f}} = \{B = \bar{A} : A \in F^{\mathrm{f}}\}$ (in particular, $\bar{F}^{\mathrm{f}}$ contains complements of all finite subsets of Ω). Then $\mathbf{P} : \bar{F}^{\mathrm{f}} \to \mathbf{N}$ and $\mathbf{P}(B)$ is decreasing if $\mathbf{P}(B)$ is r-increasing to ∞. Therefore, for example, probabilities $\mathbf{P}(E) = 100$ or 200 are rather small with respect to probabilities $\mathbf{P}(C) = 1$ or 2. An event C with probability 2 (or 3) is practically certain.

Furthermore, a probability $x \in (-\infty, 0) \cap \mathbf{Z}$ is practically negligible with respect to any probability $y \in (0, 1] \cap \mathbf{Q}$. The intuitive argument is the following. A probability $\mathbf{P}(A) \in (-\infty, 0) \cap \mathbf{Z}$ is the probability of an event A with a finite number of elements in the infinitely large population Ω. A probability $\mathbf{P}(A) \in (0, 1] \cap \mathbf{Q}$ is the probability of an event A with an infinite number of elements in the infinitely large population Ω.

Therefore, p-adics provide the possibility to split zero probability to a set of probabilities (which can be compared), $0 \to [0, 0^+)$; in particular, $(-\infty, 0) \cap \mathbf{Z} \subset [0, 0^+)$. In a similar way, p-adics provide the possibility to split unit probability to a set of probabilities (which can be compared), $1 \to (1^-, 1]$; in particular, $\mathbf{N} \subset (1^-, 1]$. The only event which is certain is the whole population Ω. Of course, events with probabilities $\mathbf{P}(A) = 2$ or 3 are practically certain. However, probabilities of realizations of such events are smaller than $\mathbf{P}(\Omega) = 1$.

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