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Three Logical Proofs: The Five-Dimensional Reality of Space-Time

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Abstract — A century and a half ago, a revolution in human thought began that has gone largely unrecognized by modern scholars: A system of non-Euclidean geometries was developed that literally changed the way that we view our world. At first, some thought that space itself was non-Euclidean and four-dimensional, but Einstein ended that 'speculation' when he declared that time was the fourth dimension. Yet our commonly perceived space is four-dimensional. Einstein unwittingly circumvented that particular revolution in thought and delayed its completion for a later day, although his work was also necessary for the completion of that revolution. That later day is now approaching. The natural progress of science has brought us back to the point where science again needs to consider the physical reality of a higher-dimensional space. Science must acknowledge the truth that space is four-dimensional and space-time is five-dimensional, as required by accepted physical theories and observations, before it can move forward with a new unified fundamental theory of physical reality.

Keywords: four-dimensional—five-dimensional—space-time—Einstein—Clifford — Kaluza — Kaluza-Klein — magnetic vector potential — electromagnetism — Yukawa potential —xpanding universe — general relativity—unification—superstrings—branes—Randall-Sundrum

Introduction

Individual scientists have been searching for evidence of a fourth dimension of space for more than a century and a half. That search subsided somewhat after Albert Einstein identified time as the fourth dimension and developed the theories of relativity. However, Theodor Kaluza added a fifth dimension to space-time in 1921. Others have contributed to this line of scientific development, but not to as high an extent. Given the fact the physicists have now developed 10- and 11-dimensional theories of reality, it would seem that the search for a fourth dimension of space would have taken on a new and significant meaning, but it has not. Yet several generally accepted scientific theories and concepts do imply the existence of a fourth spatial dimension.

On the other hand, a growing number of scientists have acknowledged and embraced the simple fact that physics needs a single fundamental theory to continue its astonishing rate of progress. A complete unification of the fundamental forces of nature has itself been a long process predating the 1970s, but that unification was made basically from the relativistic point-of-view by Einstein and a few other scientists before the 1960s. Einstein searched for a successful unification of gravity and electromagnetism for the last three decades of his life, hoping that the quantum and quantum effects would emerge from the mathematical formalisms of his unified field theory, but most other scientists shared neither his optimism nor his goal. During the 1970s, quantum physicists finally adopted Einstein's goal, but not his emphasis on a unification based upon general relativity and a continuous view of the ultimate nature of reality. Quantum theorists began their own long search for unification with the discovery of the standard model, then the electroweak force and finally the hope that gravity would eventually submit to quantum analysis. They have utterly failed to achieve this last step toward unification.

All that science can say for certain is that there are presently two theories that can claim to represent the most fundamental nature of reality: Quantum theory and relativity. Unfortunately, these two are mutually incompatible. The near complete dominance of the quantum paradigm over the last century has led most physicists to conclude that any future theory that unifies physics must be based upon a discrete quantum model rather than a continuous relativistic model. The attitude that discreteness can replace continuity at all levels of reality is problematic: It reflects a general disregard for the depth and extreme nature of the major differences between the two theories. This disregard has led scientists to speculate on the structure of reality at as small a level as the Planck length, resulting in the development of quantum loop theories and other attempts to find a quantum gravity theory. Whether the existence of a major conflict between the discrete and continuous is acknowledged or not, the fact that these two models of reality are mutually incompatible is generally minimized or belittled by many theoretical scientists who overwhelmingly assume that discreteness offers the only possible solution to the problem of unification.

Recent attempts to overcome this incompatibility, such as the supergravity, superstring and brane theories, have relied heavily upon the concept of hyper-dimensional spaces. These models have been unsuccessful, yet the overall notion of hyper-dimensionality still offers a way out of the dilemma. Einstein first rendered the notion of a higher-dimensional reality plausible in 1905, but the revolution that Einstein began when he unified three-dimensional space with time to form a four-dimensional space-time continuum has never been fully realized. In the meantime, the opposing quantum concept may have fully run its course and reached its inherent theoretical limits. The modem unification theories based upon the quantum model do not seek to rectify the fundamental differences between the quantum theory and special relativity. Quantum field theories only calculate quantum effects in the relativistic limit; they do not unify the theories at the necessary fundamental level that is often claimed. Many scientists ignore the extent and importance of the differences between continuity and the discrete and instead worry about the insignificant problems of indeterminism and counting bits of information. So the latest attempts at unification have failed utterly even though the quantum theory has been attempting to quantize gravity for several decades.

There are many levels to the hyper-dimensionality problem, many of which have not yet been explored even though the central problem of dimensionality for present day science dates back a century and a half. Science has been misled and has failed to recognize the significance of a far more fundamental revolution that began in the 1850s when Bernhard Riemann developed a generalized system of non-Euclidean geometries (Riemann, 1854). Riemann's work directly implied that space is four-dimensional as well as continuous. His new system of geometry remained relatively unknown for more than a decade and was only popularized within the scientific community in the late 1860s. Simultaneously, James Clerk Maxwell developed Michael Faraday's field concept of electromagnetism into a complete theory of electromagnetism. Whether the timing of these developments was coincidental or not, and only a careful review of historical documents can determine if the simultaneous development of these theories was truly a coincidence, the two fundamental concepts of the continuity of the electromagnetic field and the four-dimensionality of space are physically related. There are three logical proofs that this fact is true.

The first logical proof derives directly from Maxwell's electromagnetic theory and deals directly with the inability of science to sufficiently explain the nature of the vector or magnetic potential used to explain magnetic induction. The second logical proof deals with the nature of matter itself as represented by the Yukawa potential and the atomic nucleus. The Yukawa potential is normally used to explain how electrical repulsion is overcome to bind particles within the nucleus. However, the mathematical expression for the potential also matches the general shape of space-time curvature within the individual particles that combine to form the nucleus. And finally, the last proof is a more general argument dealing with the simple three-dimensional orientations of spiral galaxies relative to the Riemannian curvature of the universe as a whole. Although these proofs are independent of any particular modern hyper-dimensional theory, they are supported by Kaluza's theory of five-dimensional space-time.

Electromagnetism Speaks Up

The popular concept of a 'force field' is completely erroneous. Even in a classical sense, no force is associated with a field until a material particle or body interacts with it. Force is not a characteristic of the field alone. The interaction of the field and matter results in the force, but the interaction can also be characterized by a potential energy. The energy results from the force acting on the particle in one sense, or from the relative position of the particle in the field in another sense. What exists at any particular position in the field before the interaction takes place is called the potential. So a physical field is characterized by the potential of the field, not a force.

Gravity presents a good example for the concept of potential. Gravitational field strength decreases radially outward from the center of gravity of a material body like the earth according to the inverse square law. All points that are equidistant from the center of gravity form a surface in three-dimensional space along which the gravitational potential is constant, an equipotential surface. At each point on this surface, the surface is perpendicular to a radial line drawn from the center of gravity. A material body orbiting the earth would have a constant speed along any equipotential surface. Electricity presents another simple example. In this case, the units of potential are 'volts', a common electrical unit with which everyone is familiar. Equipotential surfaces representing specific volt measurements are a commonly accepted fact of electrical fields. The fact that an equipotential surface can be formed and that the surface is perpendicular to the radius of curvature at each and every point where they intersect is a general property of fields. From a theoretical point-of-view, equipotential surfaces must exist for all physical fields. For any field, successive equipotential surfaces form onionskin-like concentric surfaces around point charges or charged bodies.

There is a direct equivalence between electricity and magnetism and that equivalence forms the basis of the electromagnetic theory. Any physical quantities or properties of electricity correspond to similar quantities and properties for magnetism. But that equivalence has not yet been fully realized since there is no such thing as magnetic 'volts' or measurable magnetic potential. Magnetic potential has been, is now and will be in the future a mathematical entity alone, given the three-dimensionality of space. Consider a simple magnetic field, perhaps that of a bar magnetic. An equipotential surface cannot be drawn or represented visually as it can for an electric field, although magnetic field lines can still represent the field. A line perpendicular to any field line through a given point on that field line, representing the magnetic vector potential at that point, cannot be connected to neighboring points of equal potential on other field lines to form a continuous surface. In other words, an equipotential surface cannot be formed in the three-dimensional space of the magnetic field represented by the field lines. All equipotential surfaces would go through the same point on a field line in three-dimensional space, which is impossible, but no other conclusion can be reached from the given physical geometry of the magnetic field.

According to Roger Penrose, the magnetic potential is "not uniquely determined by the field F, but is fixed to within the addition of a quantity $d\Theta$ where Θ is some real scalar field." The scalar field is taken to be a purely mathematical entity, such that the magnetic potential **A** "is not a locally measurable quantity" (Penrose, 2005). The magnetic potential **A** exists, but no physical experiment can measure or otherwise determine the value of **A** plus the additional quantity $d\Theta$, so the value of **A** alone cannot be uniquely determined. In a sense then, the magnetic potential exists only at the point of intersection, not beyond that point in three-dimensional space. Magnetic potential is purely a point phenomenon in three-dimensional space no matter what its value. It is a mathematical paradox, but the paradox can be solved if a higher dimension to space is used. Any connection between a given potential on one field line and neighboring field lines must be in another dimension (orthogonal direction) other than the three normal directions of common space, in order for there to exist an equipotential surface. The 'gauge factor' $d\Theta$ mentioned by Penrose actually represents a minuscule measurement or perturbation in the fourth direction that does not otherwise affect normal three-dimensional field variations in the local environment. This fact can also be seen in the equations that are commonly used to express and model magnetic potential.

Although it cannot be described or measured in a normal three-dimensional space, the magnetic potential can be expressed mathematically, by its relationship to the field, as

$$\mathbf{B} = \nabla \otimes \mathbf{A}$$

and

$$\nabla \odot \mathbf{B} = \nabla \odot \nabla \otimes \mathbf{A} = 0,$$

where B is the magnetic field strength. In this form, the quantity **A** is known as the magnetic vector potential or just the vector potential. Since the operator

$$\nabla = (\partial/\partial x\ \mathbf{i}, \partial/\partial y\ \mathbf{j}, \partial/\partial z\ \mathbf{k}),$$

taking the curl of **A** would be the mathematical equivalent of constructing the magnetic field B point-by-point by simultaneously looking at the perpendicular components to **A** in each of the three dimensions of space. These equations may seem trivial to physicists, but they have far more physical meaning than they have been given in the normally accepted electromagnetic interpretation.

The potential **A** must be simultaneously perpendicular to all three coordinates used to represent a point in space according to these formulations. However, the only 'thing' that can be perpendicular to all three dimensions of space simultaneously would be a fourth orthogonal dimension. Therefore, changes in the magnetic potential as well as magnetic potential itself are perpendicular to all three directions at any spatial position in our normally perceived physical space. Different equipotential surfaces would still be expressed by three-dimensional equations even though they are displaced in the fourth direction because they would act like three-dimensional spaces that are parallel to or stacked on top of our common three-dimensional space in the fourth direction. Given the continuity of space, our three-dimensional material world is actually embedded in a four-dimensional space (or manifold). Bernhard Riemann's original development of the generalized formulations of non-Euclidean geometry posited that an n-dimensional space would be embedded in an $n+1$-dimensional manifold, which implies that the physical reality of our three-dimensional space (where $n=3$) requires the existence of a higher-dimensional manifold. In present theories of higher-dimensional spaces, such as the various superstring theories, several higher embedding dimensions are used, but the Riemannian mathematics used in general relativity only 'requires' one higher embedding dimension.

The fact that magnetism implies a fourth dimension is not new. William Kingdom Clifford, a British geometer, tried to express Maxwell's electromagnetic theory using a four-dimensional space model in the 1870s. Clifford is better known for offering the first translation of Riemann's *Habilitationsschrift* lecture, "On the hypotheses which lie at the bases of geometry", into English in 1873, among other things. Based on his understanding and interpretation of Riemann's geometry, Clifford claimed that what we sense as matter is nothing more than three-dimensional space curved in a fourth dimension and what we conceive as matter in motion is no more than variations in that curvature (Clifford, 1870). For having stated this, Clifford's geometrical model of space is normally regarded as a precursor to Einstein's model of space-time curvature in the general theory of relativity. Most twentieth century scholars have also concluded that Clifford never developed a theory and had no followers (Eddington, 1921; d'Abro, 1927; Bell, 1940; Jammer, 1954; Hoffman, 1972; Kilmister, 1973; Swenson, 1979), so his theoretical work is viewed in this regard as a historical footnote and no more. The mathematician and historian E.T. Bell has gone so far as to characterized Clifford's anticipation of Einstein as little more than a case of some lucky person hitting "the side of a barn at forty yards with a charge of buckshot" (Bell, 1937), but this view of history is completely false. While Clifford's physical theories have gone unnoticed, Clifford numbers and his system of bi-quaternions have found new uses in some modern interpretations of quantum theory and relativity (Power, 1970; Gurney, 1983; Chisholm and Common, 1985) even though they were originally developed to describe his four-dimensional space, a fact that should imply new ways of interpreting the quantum.

Many modern scholars have mistakenly interpreted Clifford's theoretical model of a four-dimensional space in physics against a historical mindset biased by an early twentieth century view of general relativity (Beichler, 1996). Clifford's main purpose was not to develop a new theory of gravity, as did Einstein several decades later. Clifford's original theoretical work only dealt with Maxwell's electromagnetic theory even though he planned to add gravity to his theory at a later date (Clifford, 1887), if he had not died. Actually, Clifford was developing what we would today consider a unified field theory or better yet a theory of everything. He was fond of saying that he was "solving the universe" (Pollock in Clifford, 1879), which was his way of describing a single theory that covered all of the natural forces. Clifford attempted first to explain magnetic induction, not gravity, with his four-dimensional geometry (Pearson in Clifford, 1885). Magnetic induction is governed by the equation $B = \nabla\otimes\mathbf{A}$, providing a direct link between the current logical argument for a four-dimensional space and Clifford's interpretation of Maxwell's electromagnetic induction.

Clifford published numerous mathematical papers on the motion of three-dimensional matter in four-dimensional elliptical (single polar Riemannian) spaces. He also published a book that actually presented his first step in building a proper theory, that is, for any of his peers who understood what he was trying to do. Historians and scholars today do not understand what Clifford was attempting to accomplish, so they only see the book as a simple introductory treatise on kinematics. Anyone looking for a completed gravity theory in Clifford's work simply will not find it. Nearly all modern historians have mistakenly claimed that he never published his theory because they are looking for a nonexistent gravity theory with time as a fourth dimension.

Clifford expressed the opinion that all energies are either potential or kinetic (Clifford, 1880), but he also believed that kinetic energies in three-dimensional space would become potential energies in his four-dimensional spatial framework. In other words, forces in three-dimensional space would reduce to constant variations in position along paths in a four-dimensional curved space, an idea that was made current in general relativity. However, the modern concept only deals with gravity as modeled by modem relativity theory while Clifford meant to apply the concept to all forces in his model. Upon this hypothesis, he published the first volume of a series of books titled *Elements of Dynamic* (Clifford, 1878). His first volume was subtitled *Kinematics*. Everyone that knew Clifford or his work knew that dynamics in three-dimensional space is just kinematics in Clifford's four-dimensional space, that is why he referred to his explanation of Dynamics as Kinematics in the book title. He was writing about four-dimensional kinematics, which was equivalent to three-dimensional dynamics in his mind and theoretical model. Coincidentally, this same book is recognized by historians as the first published statement by a mathematician that distinguished between the cross and dot products in vector algebra (Crowe, 1967), the same dot and cross products that are used in the vector and scalar representations of magnetic potential given above. It should be clear then that Clifford understood the four-dimensionality of magnetic potential a full century before the modem scientific community took the unification of gravity and electromagnetism seriously.

In developing his theory, Clifford faced the problem that no mathematical formalism existed to express his four-dimensional ideas. So he used a form of quaternions of his own invention (bi-quaternions) to express his four-dimensional model of space (Clifford, 1882). Unfortunately, quaternions lost favor in the late nineteenth century to vectors and their use was largely abandoned during the first few decades of the twentieth century. So no one today would even recognize that Clifford's mathematics represented his four-dimensional theory of physical reality. Einstein's theoretical work on a theory of gravity used the Levi-Civita tensor formalisms that had developed along a different line of reasoning than Clifford used for his quaternion algebra. The tensor calculus used by Einstein was only developed after Clifford's death.

As stated above, Clifford did not ignore the effect of his four-dimensional model of matter on the Newtonian theory of gravity. Clifford died of consumption in 1879 at the age of 34 and never completed his research, but it is still possible to discover what he planned to eventually accomplish with his four-dimensional model. His colleagues were so impressed with his theoretical ideas that both his published and unpublished works were collected, edited and published within a decade after his death. His followers and colleagues published everything that they could find, including lecture notes of classes that he taught, because they thought that his theoretical work was important enough to save for posterity and the future. Clifford's outline for the second volume of his *Elements of Dynamic* was among the unfinished works that were published. His student Robert Tucker edited this book. In it, Clifford stated his views on the theory of gravity and outlined how he would change gravity given his new four-dimensional geometry, thus indicating the fact that he was searching for, and may have found but never published, a unified field theory. But we will never know that fact for sure.

Of course, philosophical and mathematical arguments are not as valuable in science as observation and experimental verification. Yet there is some experimental evidence supporting the existence of magnetic potential in the Aharonov-Bohm effect (Aharonov & Bohm, 1959). In the Aharonov-Bohm experiment, an electron beam is split in such a manner that the two resulting beams pass on either side of an upright solenoid before coming back together on a screen. The solenoid is oriented in such a way that the twin beams cut across the field lines (perpendicular to B) and thus the net force acting on them is zero. Yet when the beams come together at the screen they interfere with each other. The interference clearly shows that the wave functions associated with the electron beams are out of phase, yet they should not be out of phase by the normal standards of Maxwell's electromagnetic theory. Although the effect is somewhat paradoxical, it is normally interpreted as evidence that the magnetic potential associated with the magnetic field is real even though it cannot be measured or experimentally determined. While the net force is zero, an integration of the potential A in a closed loop around the coil is not zero. The common interpretation of this experiment introduces a quantum solution (Bohm & Hiley, 1993). However, this effect can be simply explained and understood within the four-dimensional framework of electromagnetic induction. In other words, a classical electromagnetic interpretation can be used to explain the results if a physically real four-dimensional space that is associated with the magnetic vector potential is assumed.

While the net force is zero on either of the electron beams, the electrons are moving at a constant speed through different portions of the coil's magnetic field. So they each follow paths of varying potential (surfaces) in four-dimensional space corresponding to the portions of the magnetic field through which they travel. Since they are following four-dimensional paths of different lengths, they are out of phase when they reach the screen and interfere with each other. The principle is similar to a satellite orbiting the earth at a constant speed. The constant speed holds the satellite to a path along a gravitational equipotential surface. When the speed changes, the satellite follows a path through different equipotential surfaces. The orbital speed determines the altitude of the orbit and the potential path (surface) along which the satellite travels. The electrons in the beam also follow curved potential paths in the fourth dimension, which are different according to the portions of the magnetic field through which they pass in three-dimensional space. The difference in curved paths in four-dimensional space puts them out of phase at the end of the trip even though their paths in three-dimensional space, the projections of their paths in four-dimensional space, are not curved.

And finally, given a real fourth dimension of space that is characterized by magnetic potential, anything that emits a normal transverse electromagnetic wave in three-dimensional space would also cause a corresponding compressive wave of magnetic potential variation in the fourth direction of space. Numerous scientists have claimed to show the mathematical possibility of such longitudinal electromagnetic waves. Edmund T. Whittaker's model of 1903 is perhaps the best known of these attempts (Whittaker 1903, 1904). According to Whittaker,

> . . . thus we have the result, that the general solution of Laplace's equation
>
> $$\frac{M^2V}{Mx^2}+\frac{M^2V}{My^2}+\frac{M^2V}{Mz^2}=0$$
>
> is
>
> $$V=\int_0^{2\pi} f(z+ix\cos u+iy\sin u,u)du,$$
>
> where f is an arbitrary function of the two arguments
>
> $z + ix\cos u + iy\sin u$ and u.
>
> Moreover, it is clear from the proof that no generality is lost by supposing that f is a periodic function of *u* (Whittaker, 1903).

The variable *u* actually represents the fourth dimension of space while V is the magnetic potential. This interpretation renders Whittaker's formulation compatible with modem advances in the laws of electromagnetism without surrendering the possibility of a longitudinal electromagnetic wave. The function **f** is periodical with respect to u, which means that the fourth dimension is closed with respect to the other three dimensions of space. This closure corresponds completely to Kaluza's closure condition for the fifth dimension of space-time, while the factor of *du* over which the function **f** is integrated corresponds to Penrose's gauge invariance $d\Theta$.

In this respect, the fourth dimension of space is independent of the length of the extension in the fourth direction, such that the fifth direction of space-time can be either microscopic or macroscopic in extent. There is no difference between the two in the function **f** as long as the fourth dimension of space is closed. Whittaker then analyzed the general form of the differential equations for wave motion

$$\frac{M^2V}{Mx^2}+\frac{M^2V}{My^2}+\frac{M^2V}{Mz^2}=k^2\frac{M^2V}{Mt^2}$$

to demonstrate that the mathematical model can account for a longitudinal electromagnetic wave. However, if V is taken to mean the magnetic potential in the fourth direction of space, then the magnetic potential V can be related directly to the concept of proper time in special relativity. Whittaker's concept of a longitudinal component of electromagnetic waves can thus be rendered in relativistic terms, which implies that the concept is actually a wave of changing magnetic potential propagating in the fifth direction of a five-dimensional space-time continuum.

Whether or not Maxwell's electromagnetic theory requires a longitudinal wave in its classical three-dimensional interpretation is open to debate, but the existence of a fourth dimension to space would require a corresponding longitudinal wave that propagates throughout the fourth dimension relative to the normal three dimensions of space. No one has ever detected a three-dimensional longitudinal wave, but that does not mean the wave cannot be four-dimensional. After all, no one has ever detected or measured a 'magnetic-volt' of potential in three-dimensional space either, even though the potential exists in four-dimensional space.

The Yukawa Field

Modern physics also requires the existence of a fourth spatial dimension, but this time the culprit is the Yukawa potential. The Yukawa potential normally takes the form

$$V(r) = -g^2 \frac{e^{-kr}}{r}.$$

The quantity g is real. It represents the coupling constant between the meson field and the fermion with which it interacts, at least in the normal quantum interpretation. The Yukawa potential itself arises from the exchange of a massive scalar field or particle such as the pi meson or pion (Yukawa, 1935). The negative sign guarantees that the force between particles in the nucleus is always attractive.

This potential is associated with the extremely short-range strong nuclear force and it is usually only interpreted as a quantum phenomenon. The potential associated with the Yukawa field decreases exponentially, guaranteeing the short range of the Yukawa field to little more than the outer boundaries of the nucleus. It is simply assumed that the Yukawa field cannot be interpreted within a non-quantum context, yet there is no hard and fast rule that states that the Yukawa potential cannot be interpreted geometrically. Classical fields are normally interpreted geometrically, so it would seem that the Yukawa field should also have a geometrical interpretation. Even the modern view of gravity as resulting from the curvature of space-time is geometrical in nature.

According to a simple interpretation of physical laws, the field strengths of both electric and gravitational fields vary as $1/r^2$. Traditionally, this inverse square law has been interpreted as resulting from the three-dimensionality of space and thus guaranteeing that space must be three-dimensional. As trivial as this may seem, the inverse square law has been used in the past to explain the necessity of a three-dimensional space to the laws of physics (Whitrow, 1955; Abramenko, 1958; Biichel, 1963; Freeman, 1969). In other words, the inverse square law is normally thought to imply (if not prove) that space 'must be' three-dimensional. It has also been a common practice in the past to criticize higher-dimensional theories by pointing out that gravity would not work in a higher-dimensioned space because the inverse square law would not apply. However, we commonly accept the notion of a four-dimensional space-time without any alteration to the inverse square law without realizing that we do so. The fourth dimension of time is both qualitatively and quantitatively different from the normal three dimensions of space, so it does not affect the inverse square law. By the same token, there is no hard and fast rule that unequivocally requires that a fourth dimension of space would be both quantitatively and qualitatively the same as our normal three dimensions of space. In fact, given the reality of a fourth dimension of space, nature seems to have ordained that the fourth dimension is different from our normal three dimensions of space and nature rules physics instead of the other way around. So there is no valid or compelling reason to assume that a fourth spatial dimension would have any effect on the inverse square law and gravity. In fact there are reasons to believe that the opposite is true.

Many scientists have long believed that matter is electrically constituted and electricity acts according to the inverse square law. Our perception of space is dependent on the relative positions of matter in that space. So if matter is three-dimensional we sense space as three-dimensional. The three-dimensional surface curvature of a material particle or material body may be sufficient to determine the three-dimensionality of space, but the complete three-dimensionality of the particle is not necessary according to how it outwardly appears. Nor is it complete. The interior portion of a material particle could still be higher dimensional. For instance, the interior of a proton could be a physical singularity stretching into a higher fourth dimension even though the exterior surface of the proton is still curved spherically in three-dimensional space. Space could have any number of dimensions while three-dimensional matter only determines that part of the space or manifold in which the electrical field acts and reacts. Our normal senses evolved in the three-dimensional material world of nature, so they would be limited to detect only the three-dimensionality of matter even given a real fourth dimension. Since gravity acts between material particles, which are three-dimensional due to their electrical nature, it would also act three-dimensionally even if space had four or more dimensions. While it is commonly argued that space is three-dimensional because of the inverse square law, it could also be argued that we only sense three out of a greater number of dimensions because of the inverse square law by which gravity and electricity act as they do in three dimensions.

It seems that the inverse square law only guarantees the three-dimensional actions and interactions of matter, not the other way around. The forces associated with common fields act three-dimensionally and no more. The inverse square law does not guarantee that either space itself or fields in general are three-dimensional or otherwise limited to three dimensions. Fields could be higher-dimensional entities just as space could be higher dimensional even though we only sense three dimensions of space. Matter reacts with fields in three-dimensional space because matter is outwardly three-dimensional, not because fields are three-dimensional. If fields are higher dimensional, there may be field-field interactions that occur only in the higher dimensions of space and thus remain undetected in the three-dimensional material space except by their secondary effects. An effect such as quantum entanglement could be explained in this manner. When all is taken into account, neither physical fields nor space need be limited to three dimensions by either the laws of nature or logic and reason.

On the other hand, the potentials associated with fields vary as $1/r$. So a physical field associated with a particular potential has one more factor of the variable 'r' than the potential itself because fields vary as $1/r^2$. The dimensionality of the space that the field occupies is generally two greater than the exponent of the variable '*r*' in the denominator of the formula representing the potential. This logic also follows for the Yukawa potential: The variable '*r*' in the denominator reflects the three-dimensionality of the field, but there is another term with an 'r-' factor in the exponent in the numerator of the formula. The variable 'r' in the numerator of the formula could easily represent another dimension, so the Yukawa potential would require that the space occupied by the Yukawa field is four-dimensional, not three-dimensional. The exponential term e^{-kr} represents both the geometrical structure of the particle and its associated field as extended into the fourth dimension of space. The extension of a particle in the fourth direction would occur internally relative to three-dimensional space so that the part of the material particle that we sense or detect remains the three-dimensional exterior surface of the particle.

In this model of the Yukawa potential and field, the variable '*r*' in the denominator would account for the spherical shape of elementary particles and the nucleus itself. By analogy, this would indicate that the exponential term in the numerator would refer to the geometrical shape of the Yukawa field in the higher fourth dimension. If the Yukawa field conforms to the shape of an exponential curve in the higher dimension, as opposed to the spherical shape in three-dimensional space, then the fourth dimension of space is most certainly different from the other three dimensions of normal space, as noted above.

In fact, elementary particles such as protons and neutrons would be small singularities according to the general theory of relativity; or rather they would be singular at their centers. They would therefore follow curved space-time in a shape similar to a rotated exponential curve, as shown in a normal drawing of the curved metric of a singularity (see Figure 1).

So the Yukawa field would correspond to the shape of a nucleus or elementary particles predicted by relativity theory, if general relativity is taken to depict a real curvature of three-dimensional space in a higher embedding fourth

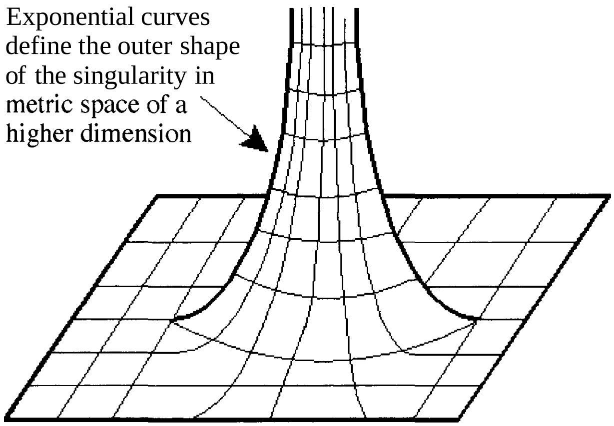

Fig. 1. The internal curvature of an elementary particle.

dimension of space. At this point, there is no need to assume a dimensionality greater than four as used in some recent theories, although there are no restrictions on space having more than four dimensions. Moreover, the curvature of space-time in general relativity is a function of the mass of a particle or body. The constant k in the Yukawa potential is also related to the mass of the exchange particle between nucleons. In both cases, the mass is related to the curvature explicit in the mathematical model, which indicates that the Yukawa potential could be modeled by the curvature of space-time as expressed by the theory of relativity rather than the particle exchange concept of quantum field theory. In either case, the Yukawa potential logically requires that space is four-dimensional and thus the space-time continuum of relativity is five-dimensional. The relationship between the Yukawa potential and general relativity leads to the third logical proof that space is four-dimensional, only this time the proof deals with the macroscopic world of the greater universe rather than the microscopic world of the quantum.

The Cosmological Connection

In the late 1920s, Edwin Hubble observed that other galaxies were receding from our Milky Way galaxy with increasing speed as the distance to the other galaxies increased. These observations indicated that our universe is expanding. Georges-Henri Lemaitre and others who developed the expansion hypothesis by a theoretical application of general relativity had already predicted the expansion. The marriage of observation and theory in this case produced one of the most spectacular successes for science in the twentieth century. The simple notion of an expanding universe is usually explained by analogy to a two-dimensional surface expanding in a third dimension.

A good example would be a balloon with spirals drawn on its surface to represent galaxies. When the balloon is blown up and expands, the spirals spread apart and move away from each other in the same pattern of motion that the receding galaxies show during astronomical observation. The expanding surface of the balloon is analogous to our expanding universe, the difference being that the balloon is a two-dimensional surface expanding outward in a third direction while the universe is a three-dimensional surface expanding into 'who knows what'. Although the phrase 'who knows what' is not an appropriate phrase for scientific use, it does represent how science views the question of what the universe is expanding into.

Some versions of modern brane theory postulate variously dimensioned branes curved in higher-dimensional bulks, so brane theorists could claim that the universe is expanding into the embedding bulks. However, brane theories have other problems to overcome: There is a discontinuity between the branes and the bulks in which they are embedded, such that the branes and bulks are separate things. As such, they break the continuity of the space-time continuum. The brane theories are based upon Klein's interpretation of Kaluza's five-dimensional theory of space-time, but they violate the basic assumptions upon which Kaluza unified electromagnetism and gravity as expressed by general relativity: Kaluza assumed the continuity of four-dimensional space-time with the fifth and higher dimension. So it would seem that the brane theories as well as the superstring theories upon which they were constructed are at odds with their own basic premise.

However, the balloon analogy gives more information about the expansion than ordinarily suspected, which implies an answer to this unanswered question about what the universe is expanding into. The spirals drawn on the balloon's surface are all rotating and expanding relative to a single point, the geometric center of the balloon, rather than any center on the surface of the balloon. This part of the analogy is often used to argue that our universe has no center within its three-dimensional expanse, which is true. The curvature of space-time in general relativity has always been considered an intrinsic property of space-time such that a higher embedding dimension has been unnecessary to explain observed and suspected phenomena. However, a higher embedding dimension, demonstrating that the curvature of space-time is an extrinsic property, is still perfectly compatible with general relativity (Misner *et al.*, 1973). Extrinsic curvature is sufficient to explain the effects of general relativity, but has never been considered necessary as long as the idea of intrinsic curvature was considered more likely. But if the concept of extrinsic curvature and a higher embedding spatial dimension does not represent our true reality, simple relativity will be violated in the case of the expanding universe and other astronomical observations.

In the balloon analogy, as stated above, the plane of rotation of the spirals and the recession of the spirals as the balloon expands are all oriented relative to a single point, the center of curvature of the balloon's surface. In the real three-dimensional spatially extended universe, all of the galaxies rotate and recede from each other at all possible angles or orientations in three-dimensional space. Yet you cannot have a mathematical property true for one configuration of spatial dimensions (two dimensions embedded in three-dimensional space) that is not true for another configuration (three dimensions embedded in a four-dimensional space). Such an inconsistency would destroy the validity of the mathematical model. The general geometric properties are the same for all spaces and embedding manifolds for an n-dimensional geometry embedded in an $n+1$-dimensional manifold. Riemannian geometry is based upon this simple idea. So, there is a logical necessity that the orientation of all of the galaxies in the expanding universe be relative to a single point or center of curvature of the universe. The natural rotations of galaxies in the universe are all relative to the same point, and the planes of galactic rotation are all tangential to the three-dimensional surface that is our space, which is perpendicular to the real extrinsic radii drawn between them and the center of a physically real curvature of our universe in a fourth spatial dimension.

In this case, it is illogical to speak of the overall curvature of the universe and then deny the reality of the higher embedding dimension because of a human sensory and perceptual bias against the possibility of a fourth spatial dimension. Perhaps local spatial curvature can be explained away as an intrinsic characteristic of the space-time continuum, but the concept of intrinsic curvature on a global level is untenable. The notion of an intrinsic radius of curvature for the whole of the universe is illogical. The three-dimensional surface of our universe is closed such that it forms a Riemannian sphere, which would require a higher embedding dimension to account for the closure. Once again, the only way to derive a direction perpendicular to all three dimensions of space simultaneously would be to adopt the geometry of a real four-dimensional embedding space. That fourth dimension or direction is orthogonal to the normal three dimensions of space. So the observed three-dimensional orientation of astronomical bodies directly requires the reality of a fourth spatial dimension. In effect, our three-dimensional universe is expanding into a fourth dimension of space. The simple fundamental notions of relative motion and actual observation, rather than any specific theory, logically require that our space is four-dimensional and thus space-time is five-dimensional.

The Kaluza Confirmation

While these logical proofs may not be completely persuasive or even persuasive enough to sway the attitudes of many within the general scientific community, there are other extenuating factors and circumstances that should be persuasive given the validity of the logical proofs. Also, these three logical proofs should be considered independent of any particular hyper-dimensional theory of space-time. They only indicate that some higher-dimensional theory would give a more correct picture of our physical reality without specifying the exact theory to be used. Yet there is already a specific scientific theory that successfully utilizes a five-dimensional space-time geometry to unify general relativity and electromagnetism: Kaluza's 1921 theory. Kaluza's theory has been largely ignored in spite of its successful derivation of Maxwell's electromagnetic theory from the general relativity of a five-dimensional space-time continuum. Most modern scientists are only familiar with Kaluza's theory through its association with the work of Oskar Klein, altering the theory to the Kaluza-Klein model of space-time. Little is known of Kaluza's original theory under these circumstances. Klein's subsequent adaptation of the theory (Klein 1926a, 1926b, 1927) was an attempt to incorporate quantum theory into the geometry of space-time. But Kaluza's theory can stand alone on its own merits, without considering Klein's extended version of the theory into the realm of the quantum. Kaluza's original theory had nothing to do with the quantum.

According to Kaluza's original theory, two mathematical conditions are necessary to unify general relativity and electromagnetic theory. All points in the four-dimensional space-time continuum are extended orthogonally into the fifth dimension along what Kaluza called A-lines. The A-lines follow circular paths in the fifth direction back to our space-time continuum, so they are closed with respect to the fifth direction. Kaluza's first condition was to close the system in the fifth direction, but the A-lines were also required to be of equal length, giving the second condition. Kaluza also suggested that the A-lines are infinitesimally short to guarantee that we could not detect the fifth dimension, although this suggestion was not a required mathematical condition. The two conditions were necessary to guarantee the mathematical consequences of adding the fifth dimension: Deriving the equations of general relativity by applying a four-transformation while obtaining the equations of electromagnetism by applying a cut-transformation.

If either of the initial conditions were to be changed or relaxed in any manner, it is possible and even likely that the results of the change would render electromagnetism and gravity incompatible if not break Kaluza's link between them altogether. But Kaluza also assumed, without so stating, a third condition of continuity in the fifth direction. Continuity was built into the calculus that Kaluza used to develop his geometrical model. So if continuity is forfeited, then Kaluza's theory could still fall apart. Before any of these conditions is changed in new extensions of Kaluza's theory, it must be shown that any of these changes, or any combination of them, does not alter Kaluza's results, the unification of gravity and electromagnetism. There are no middle roads to take here; it is all either black or white. If Kaluza's initial conditions were altered in any manner that breaks or weakens the link between gravity and electromagnetism, then the extension would be invalid for having destroyed the very foundations upon which the new theory is based. Yet changes in these conditions have been made to expedite the development of modern theories and thus could have a direct bearing on the validity of the supergravity, superstring and brane theories, all of which depend on extended versions of the Kaluza-Klein model.

When Klein adopted Kaluza's theory in an attempt to quantize the unified field, he did not relax or alter Kaluza's conditions. He merely followed Kaluza's suggestion that the extension in the fifth direction must be extremely small since we cannot detect the extra dimension. Klein equated the periodicity in the 'closed loop' condition to the quantum of action. At the time, Klein's version of the theory was largely ignored by the scientific community, which was mesmerized by other developments in quantum theory such as quantum mechanics and wave mechanics. Unfortunately, Klein could not make his theory work. He rejected his first theory and made two later attempts to rectify the errors in his theory, in 1939 and 1947 (Klein 1939, 1947), but eventually rejected his basic hypothesis and gave his theory up as a lost cause.

Klein's adaptation of Kaluza's theory, the Kaluza-Klein theory, was rediscovered in the 1970s and adopted by supergravity theorists as a method to unify gravity with the latest versions of the quantum field theories and the standard model of elementary particles. The superstring theorists adopted the Kaluza-Klein theory a few years later, but both groups of theorists have expanded the number of dimensions to 10, 11 or more. However, these scientists have never demonstrated that adding the extra dimensions above Kaluza's original five would remain consistent with the original purpose of Kaluza's theory to unify general relativity and electromagnetism. These theories are untenable and speculative and they will remain so until superstring theorists can demonstrate that adding the extra dimensions does not alter the connection between Einstein and Maxwell's theories that Kaluza's five-dimensional structure established.

On the other hand, any extension of the Kaluza-Klein theory that is superimposed on a quantum field theory should also suffer from fundamental problems because quantum field theories are by their very nature based upon a discrete model that is at odds with the assumed condition of continuity in Kaluza's original theory. Nor have the superstring theorists explained how the curvature of space-time fits into their theories, even though they take general relativity for granted as the basis of their theories. Any Kaluza or Kaluza-Klein theory that retains the infinitesimal (or Planck) extension of length in the fifth direction must deal with the same fundamental problem. The adoption of a real physical five-dimensional space-time structure, instead of a limited purely mathematical model, implies that curvature is an extrinsic characteristic of our common four-dimensional space-time continuum. However, an infinitesimally extended fifth direction seems to retain the intrinsic nature of the four-dimensional space-time by not explaining how the concept of curvature fits into the model, creating a paradox.

The superstring theories have evolved into the more general 'brane' theories. Several 'brane' theorists have speculated on all types of structures including dual three-dimensional branes, five-dimensional branes, colliding branes and curved branes within a bulk, to mention only a few examples. But it seems that they have yet to demonstrate whether these geometrical structures conform to the basic hypotheses upon which their theories depend, Kaluza's initial derivation of the general relativity and electromagnetic formulas from an extremely limited and conditional five-dimensional mathematical model of a continuous space-time. The Randall-Sundrum theory offers a case in point (Randall & Sundrum, 1999a, 1999b). In the Randall-Sundrum model, two branes are separated by a higher-dimensional bulk. One of the branes represents our common three-dimensional curved space, while gravitons traveling from our brane to the other brane are the only direct links between the branes. In one model, the second brane is an infinite distance away, effectively limiting our world to the single brane embedded in the bulk and guaranteeing a weak gravitational force. However, this model is in direct violation of Kaluza's condition that our four-dimensional world is closed with respect to the higher fifth dimension. Brane theories of this type must be required to demonstrate that their models do not disrupt the unification of electromagnetism and gravity in the Kaluza model upon which they are based. Yet no one has ever argued or even explored how such changes would affect the basic underlying principles of the original mathematical unification model developed by Kaluza.

The only theoretical research ever conducted to determine the mathematical consequences of changing Kaluza's theory only considered the relaxation of his initial suggestion of an infinitesimal extension, rather than changing any of his initial conditions. Einstein and Peter G. Bergmann completed this change in 1938 (Einstein & Bergmann, 1938). Einstein, Bergmann and Valentine Bargmann again considered it in 1941 (Einstein *et al.*, 1941). They retained the 'closed loop' and 'equal length' conditions and remained within a continuous mathematical model of five-dimensional space-time, but allowed for the possibility of macroscopically extended lengths of the A-lines. Under these conditions, they were still able to derive Maxwell's formulas and thus maintain Kaluza's unification. But Einstein eventually gave up this avenue of research toward his goal of a unified field theory because he could not justify the notion of a normal sized fifth dimension that could not be sensed or detected in any manner. Even so, Einstein listed the five-dimensional approach as one of three possibilities to develop a unified field theory in his last published book before he died (Einstein, 1956). He stipulated that the five-dimensional hypothesis would only be tenable if it could be explained why the fifth dimension cannot be detected.

Conclusion

These three logical proofs, in themselves, will not immediately change the course of science. Science has ignored the implied existence of a real fourth spatial dimension for more than a century, so it will not be so easily compelled to accept it now. However, it is not just the three logical proofs that indicate the existence of a fourth spatial dimension to our universe. It is a preponderance of the evidence that will soon force science to accept the four-dimensional reality of space. The value of these three logical proofs will only become evident over the longer term of scientific advances.

While logically proving the existence of a fourth dimension to space, these proofs also imply the geometric structure of that dimension relative to the other three. First of all, the fourth dimension of space would be different, like time, from the other three common dimensions of space. Otherwise, four-dimensionality would adversely affect the inverse square law and thus conflict with normally accepted physical laws. Instead, the fourth dimension should be characterized by changing magnetic potential except inside elementary particles where the space curvature corresponding to matter would assume the shape of an exponential curve. Both of these characteristics imply that the total extension of space in the fourth direction cannot be infinitesimally small or even microscopic as in Klein's version of Kaluza's theory. The exponentially shaped singularity at the center of elementary particles such as protons would require a non-infinitesimal extension of space in the higher dimension.

In other words, if the magnetic potential and Yukawa potential exist in nature as described, then the fourth dimension of space, or the fifth dimension of space-time, cannot be infinitesimally extended. Both logical arguments imply that the extra higher dimension is macroscopically extended as Einstein, Bergmann and Bargmann demonstrated. It is provident that Kaluza's theory has already been developed as the basis for a new unification, but the macroscopic extension in the fourth direction of space means that the present unification theories that are based upon Kaluza's suggestion and Kaluza-Klein models are not valid. The path of unification that science must follow is the path that physics and nature leads us down, not the path that some scientists decide that nature must logically follow, no matter how 'beautiful' or aesthetically pleasing those theories might be. The path that nature has decided for science is the one that leads to the four-dimensionality of space (the Clifford model) and the five-dimensionality of the space-time continuum (the Einstein-Kaluza model).

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